

On extensions of local Dirichlet forms

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Abstract Let \mathcal{E} be a Dirichlet form on $L_2(X; \mu)$ where (X, μ) is locally compact σ -compact measure space. Assume \mathcal{E} is inner regular, i.e. regular in restriction to functions of compact support, and local in the sense that $\mathcal{E}(\varphi, \psi) = 0$ for all $\varphi, \psi \in D(\mathcal{E})$ with $\varphi\psi = 0$. We construct two Dirichlet forms \mathcal{E}_m and \mathcal{E}_M such that $\mathcal{E}_m \leq \mathcal{E} \leq \mathcal{E}_M$. These forms are potentially the smallest and largest such Dirichlet forms. In particular $\mathcal{E}_m \supseteq \mathcal{E}_M$, $(\mathcal{E}_M)_m = \mathcal{E}_m$ and $(\mathcal{E}_m)_M = \mathcal{E}_M$. We analyze the family of local, inner regular, Dirichlet forms \mathcal{F} which extend \mathcal{E} and satisfy $\mathcal{E}_m \leq \mathcal{F} \leq \mathcal{E}_M$. We prove that the latter bounds are valid if and only if $\mathcal{F}_M = \mathcal{E}_M$, or $\mathcal{F}_m = \mathcal{E}_m$, or $D(\mathcal{E}_M)$ is an order ideal of $D(\mathcal{F})$. Alternatively the \mathcal{F} are characterized by $D(\mathcal{E}_M) \cap L_\infty(X)$ being an algebraic ideal of $D(\mathcal{F}) \cap L_\infty(X)$. As an application we show that if \mathcal{E} and \mathcal{F} are strongly local then the Ariyoshi–Hino set-theoretic distance is the same for each of the forms \mathcal{E} , \mathcal{E}_M and \mathcal{F} . If in addition \mathcal{E}_m is strongly local then it also defines the same distance. Finally we characterize the uniqueness condition $\mathcal{E}_M = \mathcal{E}_m$ by capacity estimates.

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1 Introduction

The standard theory of Dirichlet forms [Fuk80] [BH91] [FOT94] encompasses the quadratic forms associated with real, positive, symmetric, second-order, elliptic operators on domains in \mathbf{R}^d . The theory also extends to the non-symmetric situation both real [MR92] and complex [Ouh05]. The Dirichlet form theory, supplemented with an intrinsic distance, has led to interesting generalizations of many of the structural properties of the elliptic operators and in particular to those related to the underlying Gaussian structure (see, for example, [Mos94] [BM95] [Stu94, Stu95, Stu96, Stu98]). These results are based on the analysis of a fixed local, regular, Dirichlet form. In contrast our aim is to analyze the family of possible Dirichlet form extensions of a given local Dirichlet form. This is a partial analogue of the classification of the submarkovian extensions of an elliptic operator, i.e. the extensions which generate submarkovian semigroups. The extension problem was considered in [Fuk80], Section 2.2.3, for the Laplacian on a domain in \mathbf{R}^d and in [FOT94] for a second-order elliptic operator H with smooth coefficients. It was established that H has both a minimal and a maximal submarkovian extension (see [FOT94], Theorem 3.3.1). The minimal extension corresponds to Dirichlet boundary conditions and the maximal extension to Neumann conditions. Each submarkovian extension is determined by a Dirichlet form extension of the quadratic form h corresponding to H . These Dirichlet forms are intermediate to the forms of the maximal and minimal extensions with respect to the usual ordering of quadratic forms. The form ordering is closely related to the ordering of the corresponding submarkovian semigroups. The semigroup associated with the intermediate form dominates the semigroup corresponding to Dirichlet boundary conditions and is dominated by the Neumann semigroup (see [Ouh96] and [Ouh05], Chapters 2 and 4). The ordering is also related to the lattice ordering of the domains of the Dirichlet forms. In fact under a mild assumption of regularity the local Dirichlet forms corresponding to semigroups sandwiched between the Neumann and Dirichlet semigroups can be completely characterized by generalized Robin boundary conditions [AW03a, AW03b]. In the sequel we obtain analogues and extensions of these results for local Dirichlet forms.

Let \mathcal{E} be a local Dirichlet form which is inner regular, i.e. it satisfies the usual regularity property in restriction to the functions with compact support. We construct for each such \mathcal{E} a maximal Dirichlet form \mathcal{E}_M and a minimal form \mathcal{E}_m . The maximal form is a restriction of \mathcal{E} and the minimal form an extension of \mathcal{E}_M . But \mathcal{E}_m is not necessarily an extension of \mathcal{E} . The submarkovian semigroup S^m corresponding to \mathcal{E}_m dominates the semigroup S^M corresponding to \mathcal{E}_M . The construction of both forms is by ‘interior approximation’ and corresponds abstractly to the imposition of either Dirichlet or Neuman boundary conditions even in the absence of an explicit boundary. For a precise formulation we refer to Theorem 3.1 in Section 3. The proof of the theorem and the subsequent analysis rely on a number of structural results which are summarized in Section 2. Then in Section 4 we characterize the local, inner regular, Dirichlet form extensions \mathcal{F} of \mathcal{E} which satisfy $\mathcal{E}_m \leq \mathcal{F} \leq \mathcal{E}_M$ as the forms with $\mathcal{F}_M = \mathcal{E}_M$ or $\mathcal{F}_m = \mathcal{E}_m$. Again \mathcal{F} is an extension of \mathcal{E}_M but \mathcal{E}_m is not necessarily an extension of \mathcal{F} . The forms \mathcal{F} can also be characterized by a number of order properties. In fact $\mathcal{E}_m \leq \mathcal{F} \leq \mathcal{E}_M$ if and only if the domain $D(\mathcal{E}_M)$ of \mathcal{E}_M is an order ideal of the domain $D(\mathcal{F})$ of \mathcal{F} (see [Ouh05] Section 2.3) or if and only if $B(\mathcal{E}_M)$, the bounded functions in $D(\mathcal{E}_M)$, is an algebraic ideal of $B(\mathcal{F})$ (see Proposition 2.7). The latter algebraic property is fundamental for much of the analysis. Finally in Section 5 we discuss the set-theoretic distance associated with a strongly local Dirichlet form and show

that it has the same values for all the forms \mathcal{F} satisfying the foregoing conditions. We also discuss the uniqueness criterion $\mathcal{E}_m = \mathcal{E}_M$ and its characterization in terms of capacity conditions.

2 Preliminaries

We begin by summarizing some standard and some not so standard elements of the theory of quadratic forms which are necessary in the subsequent discussion of extensions of Dirichlet forms. We refer to [Kat66, Kat80] for a detailed description of the general theory and to [BH91] [FOT94] for the theory of Dirichlet forms. Throughout we assume that X is a locally compact σ -compact metric space and μ a positive Radon measure with $\text{supp } \mu = X$. The corresponding real L_p -spaces are denoted by $L_p(X)$.

Let \mathcal{E} be a positive, densely-defined, quadratic form on $L_2(X)$ with domain $D(\mathcal{E})$ and corresponding graph norm $\varphi \in D(\mathcal{E}) \mapsto \|\varphi\|_{D(\mathcal{E})} = (\mathcal{E}(\varphi) + \|\varphi\|_2^2)^{1/2}$. If \mathcal{E} is closed with respect to the graph norm then it is the form of a positive self-adjoint operator H , i.e. $D(\mathcal{E}) = D(H^{1/2})$ and $\mathcal{E}(\varphi) = \|H^{1/2}\varphi\|_2^2$, which generates a self-adjoint contraction semigroup S . If \mathcal{E} is closable with respect to the graph norm its closure is denoted by $\overline{\mathcal{E}}$. A subspace D of the domain $D(\mathcal{E})$ of the closable form \mathcal{E} is defined to be a core of \mathcal{E} if $\overline{\mathcal{E}} = \overline{\mathcal{E}|_D}$. Note that this definition coincides with that in [Kat66] but differs from that of [FOT94].

The form \mathcal{E} is defined to be Markovian if for each $\varphi \in D(\mathcal{E})$ one has $\varphi \wedge 1 \in D(\mathcal{E})$ and

$$\mathcal{E}(\varphi \wedge 1) \leq \mathcal{E}(\varphi) \quad (1)$$

and \mathcal{E} is defined to be Dirichlet if it is both closed and Markovian. The Markovian property extends to a much broader class of mappings. A map $F: \mathbf{R} \mapsto \mathbf{R}$ is called a normal contraction if $|F(x) - F(y)| \leq |x - y|$ and $F(0) = 0$. If \mathcal{E} is a Dirichlet form and F a normal contraction then $F \circ (D(\mathcal{E})) \subseteq D(\mathcal{E})$ and

$$\mathcal{E}(F \circ \varphi) \leq \mathcal{E}(\varphi) \quad (2)$$

for all $\varphi \in D(\mathcal{E})$.

If \mathcal{E} is a Dirichlet form the corresponding semigroup S is submarkovian, i.e. if $0 \leq \varphi \leq \mathbf{1}$ then $0 \leq S_t \varphi \leq \mathbf{1}$ for all $t > 0$. Conversely if S is a self-adjoint submarkovian semigroup then the quadratic form \mathcal{E} corresponding to its generator H is Dirichlet.

If \mathcal{E} is Dirichlet the subspace $B(\mathcal{E}) = D(\mathcal{E}) \cap L_\infty(X)$ of bounded functions in the domain $D(\mathcal{E})$ is an algebra and also a core of \mathcal{E} . (See [BH91] Sections 1.1–1.3). We let $B_c(\mathcal{E})$ denote the subalgebra of $B(\mathcal{E})$ spanned by the functions with compact support and set $C_c(\mathcal{E}) = D(\mathcal{E}) \cap C_c(X) = B_c(\mathcal{E}) \cap C(X)$.

In the sequel we consider two regularity properties for the Dirichlet form \mathcal{E} . These properties are defined with the aid of the following conditions.

1. $C_c(\mathcal{E})$ is dense in $C_0(X)$ with respect to the supremum norm,
 2. $C_c(\mathcal{E})$ is dense in $B_c(\mathcal{E})$ with respect to the graph norm,
- $$\left. \vphantom{\begin{matrix} 1. \\ 2. \end{matrix}} \right\} \quad (3)$$

where $C_0(X)$ denotes the space of continuous functions over X which vanish at infinity. (This notation again differs from that of [FOT94].) We define \mathcal{E} to be semi-regular if

Condition 1 of (3) is satisfied and inner regular if both Conditions 1 and 2 are satisfied. The standard definition of regularity (see [FOT94] Section 1.1) replaces Condition 2 with the stronger assumption that $C_c(\mathcal{E})$ is dense in $B(\mathcal{E})$ or, equivalently, dense in $D(\mathcal{E})$ but the latter assumption is not widely applicable in the analysis of extensions of Dirichlet forms. This is illustrated by the Dirichlet forms associated with the Laplacian on a domain Ω in Euclidean space \mathbf{R}^d . The form corresponding to Dirichlet boundary conditions is given by $\mathcal{E}_D(\varphi) = \|\nabla\varphi\|_2^2$ with $D(\mathcal{E}_D) = W_0^{1,2}(\Omega)$ and it is regular in the sense of [FOT94]. But the form corresponding to Neumann boundary conditions, $\mathcal{E}_N(\varphi) = \|\nabla\varphi\|_2^2$ with $D(\mathcal{E}_N) = W^{1,2}(\Omega)$, is inner regular although it is not regular unless, exceptionally, $W_0^{1,2}(\Omega) = W^{1,2}(\Omega)$. Regularity fails, for the same reason, for extensions with mixtures of Dirichlet and Neumann conditions or with Robin boundary conditions although inner regularity persists.

Semi-regularity has the following local implications.

Lemma 2.1 *Assume \mathcal{E} is semi-regular. Let Y be a bounded open subset of X .*

- I. *$D(\mathcal{E}) \cap C_c(Y)$ is dense in $C_0(Y)$ with respect to the supremum norm and in $L_2(Y)$ with respect to the L_2 -norm.*
- II. *There is an $\eta \in C_c(\mathcal{E})$ with $0 \leq \eta \leq 1$ and $\eta = 1$ on Y .*

Proof I. It suffices to establish the density in $C_0(Y)$ since this implies density in $L_2(Y)$. But each $\varphi \in C_0(Y)$ has a unique decomposition into a positive and a negative component. Therefore it suffices to prove that each positive $\varphi \in C_0(Y)$ can be approximated uniformly by a sequence of $\varphi_n \in D(\mathcal{E}) \cap C_c(Y)$. It follows, however, from the semi-regularity that for each $n \in \mathbf{N}$ there is a $\psi_n \in C_c(\mathcal{E})$ such that $\|\varphi - \psi_n\|_\infty \leq 1/(2n)$. Replacing ψ_n by $\psi_n \wedge 0$ does not affect this estimate so one can assume that ψ_n is positive. Then one must have $0 \leq \psi_n(y) \leq 1/(2n)$ for all $y \in Y^c$. Hence $\varphi_n = \psi_n - (\psi_n \wedge (1/2n)) \in D_Y(\mathcal{E})$ and $\|\varphi - \varphi_n\|_\infty < 1/n$.

II. There is a $\varphi \in C_0(X)$ such that $\varphi \geq 2$ on Y . But it follows from semi-regularity of \mathcal{E} that there is a $\psi \in C_c(\mathcal{E})$ such that $\|\varphi - \psi\|_\infty \leq 1$. Therefore $\psi \geq 1$ on Y . Then $\eta = 0 \vee (\psi \wedge 1) \in C_c(\mathcal{E})$ and $\eta = 1$ on Y . \square

One can associate with each Dirichlet form \mathcal{E} and each $\xi \in B(\mathcal{E})$ the truncated form \mathcal{E}_ξ by setting $D(\mathcal{E}_\xi) = B(\mathcal{E})$ and

$$\mathcal{E}_\xi(\varphi, \psi) = 2^{-1}(\mathcal{E}(\xi\varphi, \psi) + \mathcal{E}(\varphi, \xi\psi) - \mathcal{E}(\xi, \varphi\psi)) \quad (4)$$

for all $\varphi, \psi \in B(\mathcal{E})$. Then

$$\mathcal{E}_\xi(\varphi) = \mathcal{E}_\xi(\varphi, \varphi) = \mathcal{E}(\varphi, \xi\varphi) - 2^{-1}\mathcal{E}(\xi, \varphi^2) \quad (5)$$

for all $\varphi \in B(\mathcal{E})$. If $\xi \in B(\mathcal{E})_+$ then \mathcal{E}_ξ is a Markovian form which satisfies the bounds $0 \leq \mathcal{E}_\xi(\varphi) \leq \|\xi\|_\infty \mathcal{E}(\varphi)$ for all $\varphi \in B(\mathcal{E})$ (see [BH91], Proposition I.4.1.1). Therefore the quadratic form \mathcal{E}_ξ extends by continuity to a Markovian form on $D(\mathcal{E})$ although the identity (5) is not necessarily valid for the extension. Moreover, if \mathcal{E} is semi-regular the definition of the truncated forms can be extended to all $\xi \in C_0(X)$. Then $\xi \in C_0(X) \mapsto \mathcal{E}_\xi(\varphi)$ is a positive linear functional for each $\varphi \in D(\mathcal{E})$. Therefore there is a Radon measure μ_φ such that $\mu_\varphi(\xi) = \mathcal{E}_\xi(\varphi)$ for all $\xi \in C_0(X)$.

Although the truncated forms \mathcal{E}_ξ are Markovian they are not necessarily closed nor closable. Therefore we consider their relaxations. The relaxation of a positive quadratic form is defined as the largest positive closed form \mathcal{E}_0 which is dominated by \mathcal{E} , i.e. the largest positive closed form \mathcal{E}_0 such that $D(\mathcal{E}) \subseteq D(\mathcal{E}_0)$ and $\mathcal{E}_0(\varphi) \leq \mathcal{E}(\varphi)$ for all $\varphi \in D(\mathcal{E})$. The relaxation is also called the lower semi-continuous regularization (see [ET76], page 10) or the relaxed form (see [Dal93], page 28). Alternatively, it can be characterized as the closure of the regular part of the form (see [Sim78]). Note that $\mathcal{E}_0(\varphi) = \mathcal{E}(\varphi)$ for all $\varphi \in D(\mathcal{E})$ if and only if \mathcal{E} is closable and then $\mathcal{E}_0 = \overline{\mathcal{E}}$.

The domain $D(\mathcal{E}_0)$ of the relaxation is the subspace of $L_2(X)$ spanned by the $\varphi \in L_2(X)$ for which there is a sequence $\{\psi_n\}_{n \geq 1}$ with $\psi_n \in D(\mathcal{E})$ such that $\lim_{n \rightarrow \infty} \|\psi_n - \varphi\|_2 = 0$ and $\liminf_{n \rightarrow \infty} \mathcal{E}(\psi_n) < \infty$. Moreover,

$$\mathcal{E}_0(\varphi) = \liminf_{\psi \rightarrow \varphi} \mathcal{E}(\psi) \quad (6)$$

where $\liminf_{\psi \rightarrow \varphi} \mathcal{E}(\psi)$ indicates the infimum over the $\liminf_{n \rightarrow \infty} \mathcal{E}(\psi_n)$ for all choices of $\psi_n \in D(\mathcal{E})$ such that $\lim_{n \rightarrow \infty} \|\psi_n - \varphi\|_2 = 0$ (see [Sim77], Theorem 3). If \mathcal{E} is closable then $\mathcal{E}_0 = \overline{\mathcal{E}}$ and this characterization follows from [Kat66], Theorem VI.1.16. In analogy with the closable case we define a subspace D of $D(\mathcal{E})$ to be a core of \mathcal{E} if $\mathcal{E}_0 = (\mathcal{E}|_D)_0$.

The relaxation procedure has many properties similar to the closure. In particular if the two forms \mathcal{E} and \mathcal{F} satisfy $0 \leq \mathcal{E} \leq \mathcal{F}$ then their relaxations satisfy $0 \leq \mathcal{E}_0 \leq \mathcal{F}_0$. The following observation is used regularly.

Lemma 2.2 *The relaxation of a Markov form is a Dirichlet form.*

Proof Let \mathcal{E}_0 denote the relaxation of the Markov form \mathcal{E} . If $\varphi \in D(\mathcal{E}_0)$ then there are $\psi_n \in D(\mathcal{E})$ such that $\lim_{n \rightarrow \infty} \|\psi_n - \varphi\|_2 = 0$ and $\lim_{n \rightarrow \infty} \mathcal{E}(\psi_n) < \infty$. But since $x \mapsto x \wedge 1$ is a normal contraction (see [BH91], Section I.2.3) it follows that $\lim_{n \rightarrow \infty} \|\psi_n \wedge \mathbb{1} - \varphi \wedge \mathbb{1}\|_2 = 0$. Moreover, $\liminf_{n \rightarrow \infty} \mathcal{E}(\psi_n \wedge \mathbb{1}) \leq \liminf_{n \rightarrow \infty} \mathcal{E}(\psi_n) < \infty$ since \mathcal{E} is Markovian. This, however, implies that $\varphi \wedge \mathbb{1} \in D(\mathcal{E}_0)$ and $\mathcal{E}_0(\varphi \wedge \mathbb{1}) \leq \mathcal{E}_0(\varphi)$. Thus \mathcal{E}_0 is closed and Markovian, i.e. Dirichlet. \square

Much of the subsequent analysis is for local Dirichlet forms but there are various notions of locality. In particular the definitions of [BH91] and [FOT94] are different. We adopt definitions intermediate between these two texts. Specifically we define the Dirichlet form \mathcal{E} to be local if $\mathcal{E}(\varphi, \psi) = 0$ for all $\varphi, \psi \in D(\mathcal{E})$ such that $\varphi \psi = 0$. Further we define \mathcal{E} to be strongly local if $\mathcal{E}(\varphi, \psi) = 0$ for all $\varphi, \psi \in D(\mathcal{E})$ such that $(\varphi + a\mathbb{1})\psi = 0$ for some $a \in \mathbf{R}$. These definitions are similar to those of [FOT94], Section 1.1, where they are only required for functions of compact support. The second condition coincides exactly with the definition of locality in [BH91], Section I.1.5. In fact the locality conditions are only necessary for all pairs φ, ψ in a suitable core.

Proposition 2.3 *Let D be a core of the Dirichlet form \mathcal{E} . Assume $\varphi \in D$ implies $|\varphi| \in D$. Then the following conditions are equivalent:*

- I. (I'.) $\mathcal{E}(\varphi, \psi) = 0$ for all $\varphi, \psi \in D(\mathcal{E})$ (for all $\varphi, \psi \in D$) such that $\varphi \psi = 0$,
- II. (II'.) $\mathcal{E}(|\varphi|) = \mathcal{E}(\varphi)$ for all $\varphi \in D(\mathcal{E})$ (for all $\varphi \in D$).

Proof Clearly $I \Rightarrow I'$.

$I' \Rightarrow II'$. If $\varphi \in D$ then $|\varphi| \in D$ and $\varphi_{\pm} = (|\varphi| \pm \varphi)/2 \in D$. But $\varphi_+ \varphi_- = 0$. Therefore Condition I' gives $\mathcal{E}(|\varphi|) = \mathcal{E}(\varphi_+ + \varphi_-) = \mathcal{E}(\varphi_+) + \mathcal{E}(\varphi_-) = \mathcal{E}(\varphi_+ - \varphi_-) = \mathcal{E}(\varphi)$.

$II' \Rightarrow II$. This follows from Ancona's proof, [Anc76] Proposition 4, of the continuity of the contraction $\varphi \mapsto |\varphi|$ in the graph norm of $D(\mathcal{E})$. In fact Theorem 10 of [Anc76] establishes that all the normal contractions acting on $D(\mathcal{E})$ are continuous.

$II \Rightarrow I$. Let $\varphi, \psi \in D(\mathcal{E})_+$ with $\varphi \psi = 0$. Set $\chi = \varphi - \psi$ then $|\chi| = \varphi + \psi$ and $\mathcal{E}(|\chi|) - \mathcal{E}(\chi) = 2\mathcal{E}(\varphi, \psi)$. Therefore $\mathcal{E}(\varphi, \psi) = 0$ by Condition II. This establishes the locality condition on positive elements of $D(\mathcal{E})$. Next assume $\varphi, \psi \in D(\mathcal{E})$ with $\varphi \psi = 0$ and with positive and negative components $\varphi_{\pm}, \psi_{\pm}$. Then $\varphi_+ \varphi_- = 0$ and $\psi_+ \psi_- = 0$. Therefore $(\varphi_{\pm} \psi_{\pm})^2 = \varphi_{\pm} (\varphi \psi) \psi_{\pm} = 0$. Thus all products of the φ_{\pm} with the ψ_{\pm} are zero. Then $\mathcal{E}(\varphi, \psi) = 0$ by linearity from the previous conclusion for positive elements. \square

There is a similar characterization of strong locality but the mapping $x \mapsto |x|$ is replaced by the normal contraction $x \mapsto F(x) = |x + 1| - 1$, or by $x \mapsto F(x) = |x + \lambda| - \lambda$ with $\lambda > 0$. The details of the strongly local case are rather different to the previous one and depend on yet another characterization of strong locality (see [BH91] Section I.5 and in particular Proposition I.5.1.3 and Remark I.5.1.5).

It follows from Proposition 2.3 that if \mathcal{E} is a local, closable, Markov form whose domain is invariant under the map $\varphi \mapsto |\varphi|$ then its closure is a local Dirichlet form. It does not, however, automatically follow that a similar conclusion is valid for the relaxation of a local Markovian form which is not closable (see [Mos94], Example 6.1.1). There are, however, positive results of this type [FOT94], Theorem 3.1.2, and [ERSZ06], Proposition 2.2.

The subsequent discussion of upper and lower bounds for local Dirichlet forms depends in part on two monotonic approximation arguments.

Proposition 2.4 *Let \mathcal{E}_{α} be a monotonically increasing net of positive, closed, quadratic forms. Define the limit form \mathcal{E} by $D(\mathcal{E}) = \{\varphi : \varphi \in \bigcap_{\alpha} D(\mathcal{E}_{\alpha}), \sup_{\alpha} \mathcal{E}_{\alpha}(\varphi) < \infty\}$ and*

$$\mathcal{E}(\varphi) = \lim_{\alpha} \mathcal{E}_{\alpha}(\varphi) = \sup_{\alpha} \mathcal{E}_{\alpha}(\varphi) .$$

Then \mathcal{E} is positive and closed. If \mathcal{E} , and consequently all the \mathcal{E}_{α} , are densely-defined then the positive self-adjoint operators H_{α} corresponding to the \mathcal{E}_{α} converge in the strong resolvent sense to the positive self-adjoint operator H corresponding to \mathcal{E} .

This result was substantially established by [Kat66], Theorem VIII.3.13, but this early version assumed that the limit form was \mathcal{E} closed. Subsequently it was observed in [Rob71], Section I.2.9, that \mathcal{E} is automatically closed (see also [BR81], Lemma 5.2.13). Somewhat later Kato reached the same conclusion by quite different lower semicontinuity arguments (see [Kat80] Theorem VIII.3.13a) and Simon [Sim78] gave a third distinct proof. It also follows that if the \mathcal{E}_{α} are closable then the limit form is closable since the monotonic limit of the closures $\overline{\mathcal{E}}_{\alpha}$ is a closed extension of the monotonic limit of the \mathcal{E}_{α} .

Proposition 2.5 *Let \mathcal{E}_{α} be a monotonically decreasing net of positive, closed, densely-defined, quadratic forms. Define \mathcal{E} by $D(\mathcal{E}) = \{\varphi : \varphi \in \bigcup_{\alpha} D(\mathcal{E}_{\alpha})\}$ and*

$$\mathcal{E}(\varphi) = \lim_{\alpha} \mathcal{E}_{\alpha}(\varphi) = \inf_{\alpha} \mathcal{E}_{\alpha}(\varphi) .$$

Then \mathcal{E} is positive and densely-defined. The positive self-adjoint operators H_α corresponding to the \mathcal{E}_α converge in the strong resolvent sense to the positive self-adjoint operator H_0 corresponding to the relaxation \mathcal{E}_0 of \mathcal{E} .

A version of this result was given by Kato, [Kat66] Theorem VIII.3.11, again with the assumption that the limit form is closable. The complete statement was derived by Simon [Sim78] with a slightly different terminology.

Propositions 2.4 and 2.5 also have extensions to monotone families of non-densely defined forms, e.g. forms \mathcal{E}_α defined on subspaces $L_2(X_\alpha)$ of $L_2(X)$ (see [Sim78], Section 4). This observation will be used in the subsequent discussion of order properties of submarkovian semigroups (see Propositions 2.7, 3.3, 3.5 and 3.6). Note that in both propositions, and their extensions, the strong resolvent convergence of the semigroup generators is equivalent to the strong convergence of the semigroups S^α generated by the H_α to the semigroup S generated by the limit operator H or H_0 (see [Kat66], Theorem IX.2.16).

A measurable subset A of X is defined to have finite capacity with respect to the Dirichlet form \mathcal{E} , or finite \mathcal{E} -capacity, if there is a $\varphi \in B(\mathcal{E})$ with $\varphi = 1$ on A . In particular if \mathcal{E} is semi-regular then each bounded open subset has finite capacity. The finite capacity subspace of $D(\mathcal{E})$ is defined by

$$D_{\text{cap}}(\mathcal{E}) = \{\varphi \in B(\mathcal{E}) : \text{supp } \varphi \text{ has finite capacity} \}.$$

It is invariant under normal contractions. Set $B_{\text{cap}}(\mathcal{E}) = D_{\text{cap}}(\mathcal{E}) \cap L_\infty(X)$.

Proposition 2.6 *The subspace $B_{\text{cap}}(\mathcal{E})$ is a core of \mathcal{E} .*

Proof Since $B(\mathcal{E})$ is a core it suffices to prove that each $\varphi \in B(\mathcal{E})_+$ can be approximated in the $D(\mathcal{E})$ -graph norm by a sequence $\psi_n \in B_{\text{cap}}(\mathcal{E})$. Define φ_τ by $\varphi_\tau(x) = \tau^{-1}(\varphi(x) \wedge \tau)$ with $\tau > 0$. Then $\varphi_\tau \in B(\mathcal{E})$, $0 \leq \varphi_\tau \leq 1$ and $\mathcal{E}(\varphi_\tau) \leq \tau^{-2}\mathcal{E}(\varphi)$ by the Dirichlet property. Therefore the set $A_\tau = \{x \in X : \varphi_\tau(x) = 1\} = \{x \in X : \varphi(x) \geq \tau\}$ has finite capacity. Now set $\psi_\sigma = \varphi - \varphi \wedge \sigma^{-1}$ with $\sigma > 0$. Then $\psi_\sigma \in B(\mathcal{E})$ and $\text{supp } \psi_\sigma = A_{\sigma^{-1}}$. Therefore $\psi_\sigma \in B_{\text{cap}}(\mathcal{E})$. But $\|\varphi - \psi_\sigma\|_{D(\mathcal{E})} = \|\varphi \wedge \sigma^{-1}\|_{D(\mathcal{E})} \rightarrow 0$ as $\sigma \rightarrow \infty$ by the Dirichlet form structure (see, for example, [FOT94] Theorem 1.4.2(iv)). This establishes the required approximation. \square

Finally we consider various ordering properties of two Dirichlet forms \mathcal{E} , \mathcal{F} and the corresponding submarkovian semigroups S , T . The ordering of the semigroups, i.e. the property $0 \leq S_t\varphi \leq T_t\varphi$ for all $\varphi \in L_2(X)_+$ and $t > 0$, is related to an order ideal property of the form domains (see [Ouh05], Theorem 2.24). This observation is independent of the detailed Dirichlet structure and depends only on the positivity and L_2 -contractivity of the semigroups. But for Dirichlet forms the ordering is also equivalent to an algebraic ideal property.

First if \mathcal{E} and \mathcal{F} are Dirichlet forms then Ouhabaz defines $D(\mathcal{E})$ to be an ideal of $D(\mathcal{F})$ if $\varphi \in D(\mathcal{E})$, $\psi \in D(\mathcal{F})$ and $|\psi| \leq |\varphi|$ implies that $\psi \text{sgn } \varphi \in D(\mathcal{E})$. Alternatively $D(\mathcal{E})$ is defined to be an order ideal of $D(\mathcal{F})$ if $0 \leq \psi \leq \varphi$ with $\varphi \in D(\mathcal{E})$ and $\psi \in D(\mathcal{F})$ implies that $\psi \in D(\mathcal{E})$. These two notions are closely related and if $D(\mathcal{E}) \subseteq D(\mathcal{F})$ it follows that they are equivalent, i.e. $D(\mathcal{E})$ is an ideal of $D(\mathcal{F})$ if and only if it is an order ideal. This follows from the proof of Proposition 2.23 in [Ouh05].

Secondly, since \mathcal{E} and \mathcal{F} are Dirichlet forms $B(\mathcal{E})$ and $B(\mathcal{F})$ are algebras and $B(\mathcal{E})$ is defined to be an algebraic ideal of $B(\mathcal{F})$ if $B(\mathcal{E})B(\mathcal{F}) \subseteq B(\mathcal{E})$. This property is also related to the ordering of the corresponding semigroups S and T .

Proposition 2.7 *Let \mathcal{E} and \mathcal{F} be Dirichlet forms with $D(\mathcal{E}) \subseteq D(\mathcal{F})$ and S, T the submarkovian semigroups associated with \mathcal{E} and \mathcal{F} , respectively.*

The following conditions are equivalent:

- I. $0 \leq S_t \varphi \leq T_t \varphi$ for all $\varphi \in L_2(X)_+$ and all $t > 0$,
- II. $D(\mathcal{E})$ is an order ideal of $D(\mathcal{F})$ and $\mathcal{E}(\varphi, \psi) \geq \mathcal{F}(\varphi, \psi)$ for all $\varphi, \psi \in D(\mathcal{E})_+$,
- III. $B(\mathcal{E})$ is an algebraic ideal of $B(\mathcal{F})$ and $\mathcal{E}(\varphi, \psi) \geq \mathcal{F}(\varphi, \psi)$ for all $\varphi, \psi \in D(\mathcal{E})_+$.

Proof I \Leftrightarrow II. This equivalence follows from Corollary 2.22 and Proposition 2.23 of [Ouh05]. In particular the off-diagonal order property of the forms follows from the order property of the semigroups since

$$\mathcal{E}(\varphi, \psi) = \lim_{t \rightarrow 0} t^{-1}(\varphi, (I - S_t)\psi) \geq \lim_{t \rightarrow 0} t^{-1}(\varphi, (I - T_t)\psi) = \mathcal{F}(\varphi, \psi)$$

for all $\varphi, \psi \in D(\mathcal{E})_+$.

II \Rightarrow III. If $\varphi \in B(\mathcal{E})_+$ and $\psi \in B(\mathcal{F})_+$ then $0 \leq \varphi \psi \leq \|\psi\|_\infty \varphi \in B(\mathcal{E})$. Therefore $\varphi \psi \in B(\mathcal{E})$ by the order ideal property. This establishes the algebraic ideal property for positive functions and it then follows for general $\varphi \in B(\mathcal{E})$ and $\psi \in B(\mathcal{F})$ by decomposition into positive and negative components.

III \Rightarrow II. If $0 \leq \psi \leq \varphi$ with $\varphi \in B(\mathcal{E})$ and $\psi \in B(\mathcal{F})$ then one must prove that $\psi \in B(\mathcal{E})$.

First choose $\chi_n \in B_{\text{cap}}(\mathcal{E})$ such that $\|\chi_n - \varphi\|_{D(\mathcal{E})} \rightarrow 0$ as $n \rightarrow \infty$. This is possible by Proposition 2.6. Next the modulus map $x \rightarrow |x|$ is a normal contraction and the normal contractions acting on $D(\mathcal{E})$ are strongly continuous with respect to the $D(\mathcal{E})$ -graph norm (see [Anc76], Proposition 4 and Theorem 10). Hence $\||\chi_n| - \varphi\|_{D(\mathcal{E})} \rightarrow 0$ as $n \rightarrow \infty$. Now if $\varphi_n = |\chi_n| \wedge \varphi$ then $\varphi_n \in B_{\text{cap}}(\mathcal{E})$ since $\text{supp } \varphi_n \subseteq \text{supp } \chi_n$, $0 \leq \varphi_n \leq \varphi$. Further

$$\|\varphi_n - \varphi\|_{D(\mathcal{E})} = \|(\varphi - |\chi_n|)_+\|_{D(\mathcal{E})}$$

by the lattice relation $x \wedge y = y - (y - x)_+$. But $F(x) = x_+$ is a normal contraction. Therefore $\|(\varphi - |\chi_n|)_+\|_{D(\mathcal{E})} \rightarrow 0$ as $n \rightarrow \infty$ again by continuity of the normal contractions. Hence $\varphi_n \in B_{\text{cap}}(\mathcal{E})$, $0 \leq \varphi_n \leq \varphi$ and $\|\varphi_n - \varphi\|_{D(\mathcal{E})} \rightarrow 0$.

Secondly set $\psi_n = \varphi_n \wedge \psi$. Then $B_{\text{cap}}(\mathcal{E}) \subseteq B_{\text{cap}}(\mathcal{F})$ by the assumption $D(\mathcal{E}) \subseteq D(\mathcal{F})$. Therefore $\varphi_n \in B_{\text{cap}}(\mathcal{E}) \subseteq B_{\text{cap}}(\mathcal{F})$ and $\psi \in B(\mathcal{F})_+$ from which it follows that $\psi_n \in B_{\text{cap}}(\mathcal{F})$. Moreover, because $\varphi_n \in B_{\text{cap}}(\mathcal{E})$ one can choose $\eta_n \in B(\mathcal{E})$ with $0 \leq \eta_n \leq 1$ and $\eta_n = 1$ on $\text{supp } \varphi_n$. But $\text{supp } \psi_n \subseteq \text{supp } \varphi_n$. Therefore

$$\psi_n = \eta_n \psi_n \in B(\mathcal{E}) B(\mathcal{F}) \subseteq B(\mathcal{E})$$

by the algebraic ideal property. Then

$$\|\psi_n - \psi\|_{D(\mathcal{F})} = \|(\psi - \varphi_n)_+\|_{D(\mathcal{F})}$$

by another application of the lattice relation. Now $\|\varphi_n - \varphi\|_{D(\mathcal{E})} = \|\varphi_n - \varphi\|_{D(\mathcal{F})} \rightarrow 0$ as $n \rightarrow \infty$. Therefore $\|(\psi - \varphi_n)_+\|_{D(\mathcal{F})} \rightarrow \|(\psi - \varphi)_+\|_{D(\mathcal{F})}$ as $n \rightarrow \infty$ by continuity of the normal contractions. But $(\psi - \varphi)_+ = 0$ since $0 \leq \psi \leq \varphi$. Hence $\|(\psi - \varphi_n)_+\|_{D(\mathcal{F})} \rightarrow 0$ as $n \rightarrow \infty$. Therefore $\|\psi_n - \psi\|_{D(\mathcal{F})} \rightarrow 0$. Finally as the $\psi_n \in B(\mathcal{E})$ it follows that they form a Cauchy sequence with respect to the $D(\mathcal{E})$ -graph norm. Consequently $\psi \in B(\mathcal{E})$. \square

The assumption $D(\mathcal{E}) \subseteq D(\mathcal{F})$ in the proposition follows immediately if $\mathcal{E} \subseteq \mathcal{F}$. But in this latter case the off-diagonal bounds $\mathcal{E}(\varphi, \psi) \geq \mathcal{F}(\varphi, \psi)$ for $\varphi, \psi \in D(\mathcal{E})_+$ are obvious. Alternatively $D(\mathcal{E}) \subseteq D(\mathcal{F})$ is a consequence of the order relation $\mathcal{F} \leq \mathcal{E}$ but the off-diagonal bounds are not immediate from this latter relation. In conclusion we note that the foregoing proposition remains valid if \mathcal{E} is not densely-defined but is a Dirichlet form on a closed subspace of $L_2(X)$ (see [Ouh05], Section 2.6).

3 Extremal forms

Let \mathcal{E} denote a local inner regular Dirichlet form on $L_2(X)$. In this section we construct two auxiliary Dirichlet forms \mathcal{E}_m and \mathcal{E}_M with the property that $\mathcal{E}_m \leq \mathcal{E} \leq \mathcal{E}_M$. The construction is such that the forms are extremal in a sense to be established in the following section. Although the principal results require inner regularity of \mathcal{E} the construction of the forms \mathcal{E}_m , \mathcal{E}_M and a number of the intermediate statements only require semi-regularity.

First define \mathcal{E}_M as the closure of the restriction of \mathcal{E} to $C_c(\mathcal{E})$. Then, by semi-regularity of \mathcal{E} , the form \mathcal{E}_M is densely-defined. It is automatically a regular Dirichlet form which bounds \mathcal{E} from above, i.e. $\mathcal{E} \leq \mathcal{E}_M$. Note that $C_c(\mathcal{E}_M) = C_c(\mathcal{E})$ and $\mathcal{E} = \mathcal{E}_M$ if and only if \mathcal{E} is regular. It also follows from inner regularity that \mathcal{E}_M is the closure of \mathcal{E} restricted to $B_c(\mathcal{E})$. In fact the converse is valid. The closure of the restriction of \mathcal{E} to $C_c(\mathcal{E})$ is equal to the closure of the restriction to $B_c(\mathcal{E})$ if and only if \mathcal{E} is inner regular.

Secondly, to construct the lower bound \mathcal{E}_m we introduce the family \mathcal{Y} of bounded open subsets Y of X . The family \mathcal{Y} becomes a directed set when the subsets Y are ordered by inclusion. For each $Y \in \mathcal{Y}$ define the convex subset $\mathcal{C}_Y(\mathcal{E})$ of $C_c(\mathcal{E})$ by

$$\mathcal{C}_Y(\mathcal{E}) = \{\xi \in C_c(\mathcal{E}) : 0 \leq \xi \leq \mathbb{1}_Y\}.$$

Then $\mathcal{C}_Y(\mathcal{E})$ is also a directed set with respect to the natural order, e.g. if $\xi_1, \xi_2 \in \mathcal{C}_Y(\mathcal{E})$ then $\xi_{12} = \xi_1 \xi_2 - \xi_1 - \xi_2 \in \mathcal{C}_Y(\mathcal{E})$ is a common upper bound. But the truncated forms $\{\mathcal{E}_\xi\}_{\xi \in \mathcal{C}_Y(\mathcal{E})}$ form a monotonically increasing net of Markovian forms which are uniformly bounded by \mathcal{E} . Therefore one can define a Markovian form $\mathcal{E}_{m,Y}$ by $D(\mathcal{E}_{m,Y}) = B(\mathcal{E})$ and

$$\mathcal{E}_{m,Y}(\varphi) = \lim_{\xi \in \mathcal{C}_Y(\mathcal{E})} \mathcal{E}_\xi(\varphi) = \sup_{\xi \in \mathcal{C}_Y(\mathcal{E})} \mathcal{E}_\xi(\varphi) \quad (7)$$

for all $\varphi \in B(\mathcal{E})$. Since $\mathcal{E}_\xi \leq \mathcal{E}$ for all $\xi \in \mathcal{C}_Y(\mathcal{E})$ one has $\mathcal{E}_{m,Y} \leq \mathcal{E}$ and $\mathcal{E}_{m,Y}$ extends to $D(\mathcal{E})$ by continuity. Although the forms $\mathcal{E}_{m,Y}$ are Markovian they are not necessarily closed nor even closable. Nevertheless their relaxations $\mathcal{E}_{m,Y;0}$ are Dirichlet forms on $L_2(X)$ by Lemma 2.2. Next $\mathcal{E}_\xi \leq \mathcal{E}_\eta$ for $0 \leq \xi \leq \eta$ and $\mathcal{C}_{Y_1}(\mathcal{E}) \subseteq \mathcal{C}_{Y_2}(\mathcal{E})$ for all $Y_1, Y_2 \in \mathcal{Y}$ with $Y_1 \subseteq Y_2$. Therefore $\mathcal{E}_{m,Y_1} \leq \mathcal{E}_{m,Y_2} \leq \mathcal{E}$. Consequently $\mathcal{E}_{m,Y_1;0} \leq \mathcal{E}_{m,Y_2;0} \leq \mathcal{E}$. Thus the relaxations $\{\mathcal{E}_{m,Y;0}\}_{Y \in \mathcal{Y}}$ are a uniformly bounded monotonically increasing net of Dirichlet forms. One can then define a Dirichlet form \mathcal{E}_m on $L_2(X)$ by

$$D(\mathcal{E}_m) = \{\varphi \in \bigcap_{Y \in \mathcal{Y}} D(\mathcal{E}_{m,Y;0}) : \sup_{Y \in \mathcal{Y}} \mathcal{E}_{m,Y;0}(\varphi) < \infty\} \quad (8)$$

and

$$\mathcal{E}_m(\varphi) = \lim_{Y \in \mathcal{Y}} \mathcal{E}_{m,Y;0}(\varphi) = \sup_{Y \in \mathcal{Y}} \mathcal{E}_{m,Y;0}(\varphi) \quad (9)$$

for all $\varphi \in D(\mathcal{E}_m)$. It follows immediately that $D(\mathcal{E}) \subseteq D(\mathcal{E}_m)$ and $\mathcal{E}_m \leq \mathcal{E}$. Note, however, that \mathcal{E}_m is not necessarily an extension of \mathcal{E} .

Locality of \mathcal{E} ensures that each $\mathcal{E}_{m,Y}$ is also local. Moreover, $\mathcal{E}_{m,Y}$ is localized on Y . In particular if $\varphi \in B(\mathcal{E})$ and $\text{supp } \varphi \in Y^c$ then $\xi \varphi = 0$ for all $\xi \in \mathcal{C}_Y(\mathcal{E})$ and it follows that $\mathcal{E}_{m,Y}(\varphi) = 0$. Moreover, if $\varphi, \psi \in B(\mathcal{E})$ and $\varphi|_Y = \psi|_Y$ then it follows from locality that $\mathcal{E}_{m,Y}(\varphi) = \mathcal{E}_{m,Y}(\psi)$.

As an illustration let Ω be a bounded open subset \mathbf{R}^d and consider the Dirichlet form \mathcal{E} defined as the closure of $\|\nabla \varphi\|_2^2$ on the domain $\{\varphi|_\Omega : \varphi \in C_c^\infty(\mathbf{R}^d \setminus F)\}$ where F is a closed subset of the boundary $\partial\Omega$. The corresponding self-adjoint operator is the version of the Laplacian on $L_2(\Omega)$ with Dirichlet boundary conditions on F and Neumann on $\partial\Omega \setminus F$. It follows immediately from the foregoing construction that $\mathcal{E}_M(\varphi) = \|\nabla \varphi\|_2^2$ with domain $W_0^{1,2}(\Omega)$ and $\mathcal{E}_m (= \mathcal{E}_{m,\Omega})$ is the extension of \mathcal{E}_M to the domain $W^{1,2}(\Omega)$. The maximal form corresponds to Dirichlet conditions on $\partial\Omega$ and the minimal form to Neumann conditions.

Theorem 3.1 *Assume \mathcal{E} is a local, inner regular, Dirichlet form. Then the following following properties are valid:*

- I. $(\mathcal{E}_M)_m = \mathcal{E}_m$,
- II. $(\mathcal{E}_m)_M = \mathcal{E}_M$,
- III. $\mathcal{E}_m \supseteq \mathcal{E}_M$,
- IV. $B_c(\mathcal{E}) = B_c(\mathcal{E}_m) = B_c(\mathcal{E}_M)$,
- V. $C_c(\mathcal{E}) = C_c(\mathcal{E}_m) = C_c(\mathcal{E}_M)$,
- VI. \mathcal{E}_M is regular and \mathcal{E}_m is inner regular.

Proof I. Fix $Y \in \mathcal{Y}$. One can choose $\eta \in C_c(\mathcal{E})$ with $0 \leq \eta \leq 1$ and $\eta = 1$ on Y by Lemma 2.1.II. Then if $\psi \in B(\mathcal{E})$ one has $\eta\psi \in B(\mathcal{E}_M)$. Moreover,

$$(\mathcal{E}_M)_{m,Y}(\eta\psi) = \sup_{\xi \in \mathcal{C}_Y(\mathcal{E}_M)} (\mathcal{E}_M)_\xi(\eta\psi) = \sup_{\xi \in \mathcal{C}_Y(\mathcal{E})} \mathcal{E}_\xi(\eta\psi) = \sup_{\xi \in \mathcal{C}_Y(\mathcal{E})} \mathcal{E}_\xi(\psi) = \mathcal{E}_{m,Y}(\psi)$$

where the third step uses locality. Next replacing η by a net $\eta_Z \in \mathcal{C}_Z(\mathcal{E})$ with $\eta_Z = 1$ on Y and $\lim_{Z \in \mathcal{Y}} \eta_Z = 1$ pointwise one deduces that

$$\liminf_{Z \in \mathcal{Y}} (\mathcal{E}_M)_{m,Y}(\eta_Z \psi) = \mathcal{E}_{m,Y}(\psi) < \infty$$

for all $\psi \in B(\mathcal{E})$. But $\lim_{Z \in \mathcal{Y}} \|\eta_Z \psi - \psi\|_2 = 0$. Therefore ψ is in the domain of the relaxation $(\mathcal{E}_M)_{m,Y;0}$ and

$$(\mathcal{E}_M)_{m,Y;0}(\psi) = \liminf_{\varphi \rightarrow \psi} (\mathcal{E}_M)_{m,Y}(\varphi) \leq \liminf_{Z \in \mathcal{Y}} (\mathcal{E}_M)_{m,Y}(\eta_Z \psi) = \mathcal{E}_{m,Y}(\psi)$$

for all $\psi \in B(\mathcal{E})$. But $\mathcal{E}_{m,Y} \leq \mathcal{E}$ and $B(\mathcal{E})$ is a core of \mathcal{E} . Therefore the inequality $(\mathcal{E}_M)_{m,Y;0}(\psi) \leq \mathcal{E}_{m,Y}(\psi)$ extends to all $\psi \in D(\mathcal{E}) = D(\mathcal{E}_{m,Y})$ by continuity. Consequently $(\mathcal{E}_M)_{m,Y;0} \leq \mathcal{E}_{m,Y;0}$. Conversely $\mathcal{E} \leq \mathcal{E}_M$ and $\mathcal{E}_\xi \leq (\mathcal{E}_M)_\xi$ for all $\xi \in \mathcal{C}_Y(\mathcal{E}) = \mathcal{C}_Y(\mathcal{E}_M)$. Therefore $\mathcal{E}_{m,Y} \leq (\mathcal{E}_M)_{m,Y}$ and correspondingly $\mathcal{E}_{m,Y;0} \leq (\mathcal{E}_M)_{m,Y;0}$. Combining these conclusions one has

$$\mathcal{E}_{m,Y;0} = (\mathcal{E}_M)_{m,Y;0} \tag{10}$$

for all $Y \in \mathcal{Y}$. Then taking the limit over $Y \in \mathcal{Y}$ one obtains the first statement of the theorem, $\mathcal{E}_m = (\mathcal{E}_M)_m$.

Remark 3.2 The identity (10) establishes that \mathcal{E}_M is a core of $\mathcal{E}_{m,Y}$. Specifically (10) states that the relaxation of $\mathcal{E}_{m,Y}$ restricted to $D(\mathcal{E}_M)$ is equal to $\mathcal{E}_{m,Y;0}$. Then each core of \mathcal{E}_M is a core of $\mathcal{E}_{m,Y}$.

II and III. First note that $\mathcal{E}_{m,Y}(\varphi) = \mathcal{E}(\varphi) = \mathcal{E}_M(\varphi)$ for all $\varphi \in \mathcal{C}_Y(\mathcal{E})$. But it is not clear that this identity extends to the relaxation $\mathcal{E}_{m,Y;0}$. It does, however, for φ with support strictly in the interior of Y .

Let $\varphi \in \mathcal{C}_Z(\mathcal{E})$ where $Z \in \mathcal{Y}$ is such that $\overline{Z} \subset Y$. Then $\mathcal{E}_{m,Y;0}(\varphi) = \liminf_{\psi \rightarrow \varphi} \mathcal{E}_{m,Y}(\psi)$. In particular $\psi = \{\psi_n\}_{n \geq 1}$ is a sequence of $\psi_n \in D(\mathcal{E})$ such that $\|\psi_n - \varphi\|_2 \rightarrow 0$ and $\liminf_{n \rightarrow \infty} \mathcal{E}_{m,Y}(\psi_n) < \infty$. Now one can choose $\eta \in \mathcal{C}_Y(\mathcal{E})$ with $0 \leq \eta \leq 1$ and $\eta = 1$ on Z . Then $\eta \psi_n \in \mathcal{C}_Y(\mathcal{E})$. Moreover, since $\eta \varphi = \varphi$ it follows that $\|\eta \psi_n - \varphi\|_2 \rightarrow 0$. But one again deduces by locality that

$$\mathcal{E}_{m,Y}(\eta \psi_n) = \sup_{\xi \in \mathcal{C}_Y(\mathcal{E})} \mathcal{E}_\xi(\eta \psi_n) = \sup_{\xi \in \mathcal{C}_Y(\mathcal{E})} \mathcal{E}_\xi(\psi_n) = \mathcal{E}_{m,Y}(\psi_n).$$

Therefore $\liminf_{n \rightarrow \infty} \mathcal{E}_{m,Y}(\eta \psi_n) = \liminf_{n \rightarrow \infty} \mathcal{E}_{m,Y}(\psi_n) < \infty$. One then has

$$\begin{aligned} \mathcal{E}_{m,Y;0}(\varphi) &= \liminf_{\psi \rightarrow \varphi} \mathcal{E}_{m,Y}(\psi) = \liminf_{\psi \rightarrow \varphi} \mathcal{E}_{m,Y}(\eta \psi) \\ &= \liminf_{\psi \rightarrow \varphi} \mathcal{E}_M(\eta \psi) \geq \liminf_{\psi \rightarrow \varphi} \mathcal{E}_M(\psi) = \mathcal{E}_M(\varphi) \end{aligned}$$

where the third equality follows because the $\eta \psi_n \in \mathcal{C}_Y(\mathcal{E})$. Consequently $\mathcal{E}_m(\varphi) \geq \mathcal{E}_M(\varphi)$ for all $\varphi \in \mathcal{C}_Z(\mathcal{E})$ and all $Z \in \mathcal{Y}$. But $\mathcal{E}_m \leq \mathcal{E}_M$ so this latter relation must be an equality. Therefore $\mathcal{E}_m(\varphi) = \mathcal{E}_M(\varphi)$ for all $\varphi \in C_c(\mathcal{E}_M)$ and by closure for all $\varphi \in D(\mathcal{E}_M)$. Hence $\mathcal{E}_m \supseteq \mathcal{E}_M$ and $(\mathcal{E}_m)_M = \mathcal{E}_M$.

IV. The proof of this statement is somewhat longer but it is the key to proving the remaining statements. It depends on the order properties of the Markovian semigroups associated with the various Dirichlet forms.

First we introduce forms $\{\mathcal{E}_{M,Y}\}_{Y \in \mathcal{Y}}$ analogous to the forms defined in the construction \mathcal{E}_m . For each $Y \in \mathcal{Y}$ let $C_Y(\mathcal{E}) = D(\mathcal{E}) \cap C_c(Y)$. Then define $\mathcal{E}_{M,Y}$ as the closure of the restriction of \mathcal{E} to $C_Y(\mathcal{E})$. Note that $C_Y(\mathcal{E})$ is dense in $L_2(Y)$ by Lemma 2.1.I. Therefore $\mathcal{E}_{M,Y}$ is a densely defined regular Dirichlet form on $L_2(Y)$.

Secondly let $(H_{M,Y}, S^{M,Y})$ denote the operator and semigroup on $L_2(Y)$ canonically associated with $\mathcal{E}_{M,Y}$ on $L_2(Y)$. Similarly let (H_M, S^M) and (H, S) denote the operator and semigroup on $L_2(X)$ associated with \mathcal{E}_M and \mathcal{E} , respectively. Next we adopt the convention that each bounded operator B on the subspace $L_2(Y)$ can be extended to a bounded operator on $L_2(X)$, still denoted by B , through the definition $\varphi \in L_2(X) \mapsto B(\mathbb{1}_Y \varphi) \in L_2(Y)$. In particular the resolvents $(\lambda I + H_{M,Y})^{-1}$ and the semigroups $S^{M,Y}$ extend from $L_2(Y)$ to $L_2(X)$. The extended semigroups $S^{M,Y}$ are strongly continuous on $L_2(X)$ but $S_t^{M,Y} \rightarrow \mathbb{1}_Y$ as $t \rightarrow 0$. In the sequel we use the observation that Propositions 2.4 and 2.5 are valid for the semigroups $S^{M,Y}$ extended to $L_2(X)$ (see, for example, [Sim78] or [Ouh05], Section 2.6).

Next we consider the order relations for the extended semigroups on $L_2(X)$.

Proposition 3.3 *If \mathcal{E} is a local, semi-regular, Dirichlet form and $Y_1, Y_2 \in \mathcal{Y}$ with $Y_1 \subseteq Y_2$ then*

$$0 \leq S_t^{M,Y_1} \varphi \leq S_t^{M,Y_2} \varphi \leq S_t^M \varphi \quad (11)$$

for all $\varphi \in L_2(X)_+$ and all $t > 0$. Moreover, the monotonically increasing net of semigroups $\{S^{M,Y}\}_{Y \in \mathcal{Y}}$ converges strongly to S^M .

If, in addition, \mathcal{E} is inner regular then

$$0 \leq S_t^M \varphi \leq S_t \varphi \quad (12)$$

for all $\varphi \in L_2(X)_+$ and all $t > 0$.

Proof The ordering of the $S^{M,Y}$ is proved by a refinement of the argument of [ER09]. In particular it depends on the following variation of Lemma 2.2 in this reference.

Lemma 3.4 Assume \mathcal{E} is a local, semi-regular, Dirichlet form. If $\varphi \in D(\mathcal{E}_{M,Y_1})$, $\psi \in D(\mathcal{E}_{M,Y_2})_+$ where $Y_1 \subseteq Y_2$ and if

$$(\chi, \varphi) + \mathcal{E}_{M,Y_1}(\chi, \varphi) \leq (\chi, \psi) + \mathcal{E}_{M,Y_2}(\chi, \psi) \quad (13)$$

for all $\chi \in D(\mathcal{E}_{M,Y_1})_+$ then $\varphi \leq \psi$.

Proof Since $\varphi \in D(\mathcal{E}_{M,Y_1})$ and $\psi \in D(\mathcal{E}_{M,Y_2})$ there exist sequences $\varphi_n \in C_{Y_1}(\mathcal{E})$ and $\psi_m \in C_{Y_2}(\mathcal{E})$ with $\lim \|\varphi_n - \varphi\|_{D(\mathcal{E})} = 0$ and $\lim \|\psi_m - \psi\|_{D(\mathcal{E})} = 0$. But $\psi \geq 0$ and since the modulus map $\psi_m \mapsto |\psi_m|$ is continuous in the $D(\mathcal{E})$ -graph norm it follows that $\lim \| |\psi_m| - \psi \|_{D(\mathcal{E})} = 0$. Therefore one may assume that the $\psi_m \in C_{Y_2}(\mathcal{E})_+$. Then, however, $\text{supp}(\varphi_n - \psi_m)_+ \subseteq \text{supp} \varphi_n \subset Y_1$ since $\psi_m \geq 0$. So $(\varphi_n - \psi_m)_+ \in D(\mathcal{E}_{M,Y_1})$ for all n, m . Moreover, $\lim \|(\varphi_n - \psi_m)_+ - (\varphi - \psi)_+\|_{D(\mathcal{E})} = 0$. Hence $(\varphi - \psi)_+ \in D(\mathcal{E}_{M,Y_1})$.

Secondly, set $\chi = (\varphi - \psi)_+$ in (13). Then one deduces that

$$\|(\varphi - \psi)_+\|_2^2 = ((\varphi - \psi)_+, \varphi - \psi) \leq -\mathcal{E}((\varphi - \psi)_+, \varphi - \psi) = -\mathcal{E}((\varphi - \psi)_+) \leq 0,$$

where we used locality of \mathcal{E} in the last equality. Hence $(\varphi - \psi)_+ = 0$ or, equivalently, $\varphi \leq \psi$. \square

The first ordering property of Proposition 3.3 now follows as in [ER09].

Fix $\eta \in L_2(X)_+$. Then set $\varphi = (I + H_{M,Y_1})^{-1}\eta$ and $\psi = (I + H_{M,Y_2})^{-1}\eta$. Hence $\varphi \in D(\mathcal{E}_{M,Y_1})$ and $\psi \in D(\mathcal{E}_{M,Y_2})_+$. Moreover, (13) is satisfied. Therefore $(I + H_{M,Y_1})^{-1}\eta \leq (I + H_{M,Y_2})^{-1}\eta$. By rescaling and iterating one concludes that $(I + t H_{M,Y_1}/n)^{-n}\eta \leq (I + t H_{M,Y_2}/n)^{-n}\eta$ for all $t > 0$ and n . Therefore $S_t^{M,Y_1}\eta \leq S_t^{M,Y_2}\eta$ for all $t > 0$ by the Trotter product formula.

The second order property of the proposition follows from the convergence statement which is established as follows.

The net of submarkovian semigroups $\{S^{M,Y}\}_{Y \in \mathcal{Y}}$ is monotonically increasing by the first statement. It then follows as a corollary of an old result of Vigier (see [RSN55], page 261) that the net converges strongly on $L_2(X)$ to a submarkovian semigroup T . In fact the $S^{M,Y}$ converge strongly to T on each of the spaces $L_p(X)$, $p \in [1, \infty)$. The L_p -convergence follows from results of Karlin [Kar59] and Krasnoselski [Kra64] (see [KR81] Propositions A3 and A4). It remains to identify T and S^M .

Let H denote the generator of T and \mathcal{F} the Dirichlet form corresponding to H . The net of forms $\{\mathcal{E}_{M,Y}\}_{Y \in \mathcal{Y}}$ is monotonically decreasing. Hence it follows from Proposition 2.5 that \mathcal{F} is the relaxation of the limit form $\mathcal{E}_{M,X}$ defined by $D(\mathcal{E}_{M,X}) = \bigcup_{Y \in \mathcal{Y}} D(\mathcal{E}_{M,Y})$ and

$$\mathcal{E}_{M,X}(\varphi) = \lim_{Y \in \mathcal{Y}} \mathcal{E}_{M,Y}(\varphi)$$

for all $\varphi \in D(\mathcal{E}_{M,X})$. But $D(\mathcal{E}_{M,X}) \supseteq C_c(\mathcal{E})$ and $\mathcal{E}_{M,X}(\varphi) = \mathcal{E}_M(\varphi)$ for all $\varphi \in C_c(\mathcal{E})$ and therefore by continuity for all $\varphi \in D(\mathcal{E}_M)$. Hence $\mathcal{F} \leq \mathcal{E}_M$.

Secondly, since the net of semigroups $\{S^{M,Y}\}_{Y \in \mathcal{Y}}$ converges strongly to T the corresponding net of generators $Y \in \mathcal{Y} \mapsto H_{M,Y}$ converges in the strong resolvent sense to H . But $\varphi \in L_2(X) \mapsto (I + H_{M,Y})^{-1} \mathbb{1}_Y \varphi \in D(H_{M,Y}) \subseteq D(\mathcal{E}_{M,Y}) \subseteq D(\mathcal{E}_M)$ and

$$\|(I + H_{M,Y})^{-1} \mathbb{1}_Y \varphi\|_{D(\mathcal{E}_M)}^2 = (\mathbb{1}_Y \varphi, (I + H_{M,Y})^{-1} \mathbb{1}_Y \varphi) \leq \|\varphi\|_2^2.$$

Therefore it follows from the Banach–Alaoglu theorem (see, for example, [Ouh05], Lemma 1.32, or [MR92], Lemma I.2.12) that $(I + H)^{-1} \varphi \in D(\mathcal{E}_M)$ and

$$\begin{aligned} \|(I + H)^{-1} \varphi\|_{D(\mathcal{E}_M)}^2 &\leq \liminf_{Y \in \mathcal{Y}} \|(I + H_{M,Y})^{-1} \mathbb{1}_Y \varphi\|_{D(\mathcal{E}_M)}^2 \\ &= (\varphi, (I + H)^{-1} \varphi) = \|(I + H)^{-1} \varphi\|_{D(\mathcal{F})}^2. \end{aligned}$$

But this immediately implies that $\mathcal{E}_M(\psi) \leq \mathcal{F}(\psi)$ for all $\psi \in D(H)$. Since $D(H)$ is a core of \mathcal{F} one then deduces that $\mathcal{E}_M \leq \mathcal{F}$. As we have already established that $\mathcal{F} \leq \mathcal{E}_M$ it follows that the two forms are equal. Therefore $T = S^M$ and one concludes that $\{S^{M,Y}\}_{Y \in \mathcal{Y}}$ converges strongly to S^M . Since the net of semigroups is monotonically increasing one immediately deduces that $0 \leq S^{M,Y} \varphi \leq S_t^M \varphi$ for all $Y \in \mathcal{Y}$, $\varphi \in L_2(X)_+$ and $t \geq 0$.

It remains to establish the domination property (12) for inner regular \mathcal{E} . But \mathcal{E}_M is the closure with respect to the graph norm $\|\cdot\|_{D(\mathcal{E})}$ of \mathcal{E} restricted to $C_c(\mathcal{E})$. Then by inner regularity it is the closure of \mathcal{E} restricted to $B_c(\mathcal{E})$ or, equivalently, to the subspace $D_c(\mathcal{E})$ of $D(\mathcal{E})$ spanned by the functions with compact support. Now if $\varphi \in D(\mathcal{E}_M)$, $\psi \in D(\mathcal{E})_+$ and

$$(\chi, \varphi) + \mathcal{E}_M(\chi, \varphi) \leq (\chi, \psi) + \mathcal{E}(\chi, \psi)$$

for all $\chi \in D(\mathcal{E}_M)_+$ then $\varphi \leq \psi$ by Lemma 2.2 of [ER09]. But then (12) follows exactly as in the proof of Proposition 2.1 in [ER09]. \square

The next proposition gives an ordering of the semigroups $S^{m,Y;0}$ and $S^{M,Y}$ corresponding to the forms $\mathcal{E}_{m,Y;0}$ and $\mathcal{E}_{M,Y}$, respectively. The proof is a variation of the argument used to prove Proposition 3.4 in [ER09].

Proposition 3.5 *If \mathcal{E} is a local, inner regular, Dirichlet form and $Y \in \mathcal{Y}$ then*

$$0 \leq S_t^{M,Y} \varphi \leq S_t^{m,Y;0} \varphi \tag{14}$$

for all $\varphi \in L_2(X)_+$ and all $t > 0$.

Proof Define the family of forms $\{\mathcal{E}_{m,Y;\varepsilon}\}_{\varepsilon > 0}$ on the common domain $D(\mathcal{E})$ by $\mathcal{E}_{m,Y;\varepsilon} = \mathcal{E}_{m,Y} + \varepsilon \mathcal{E}$. Since $\mathcal{E}_{m,Y} \leq \mathcal{E}$ it follows that $\varepsilon \mathcal{E} \leq \mathcal{E}_{m,Y;\varepsilon} \leq (1 + \varepsilon) \mathcal{E}$. Therefore the forms $\mathcal{E}_{m,Y;\varepsilon}$ are closed on $D(\mathcal{E})$. Moreover, the $\mathcal{E}_{m,Y;\varepsilon}$ are local because the $\mathcal{E}_{m,Y}$ inherit locality from \mathcal{E} . Then since $B_c(\mathcal{E}_{m,Y;\varepsilon}) = B_c(\mathcal{E})$ the $\mathcal{E}_{m,Y;\varepsilon}$ are inner regular Dirichlet forms. But

$$C_Y(\mathcal{E}_{m,Y;\varepsilon}) = D(\mathcal{E}_{m,Y;\varepsilon}) \cap C_c(Y) = D(\mathcal{E}) \cap C_c(Y) = C_Y(\mathcal{E}).$$

Hence $(\mathcal{E}_{m,Y;\varepsilon})_{M,Y} = (1 + \varepsilon) \mathcal{E}_{M,Y}$. Next, for brevity, let T denote the submarkovian semigroup $S^{m,Y;\varepsilon}$ associated with the Dirichlet form $\mathcal{E}_{m,Y;\varepsilon}$. Then the foregoing identification implies that $S_{(1+\varepsilon)t}^{M,Y} = T_t^{M,Y}$ for all $t \geq 0$. Since the $\mathcal{E}_{m,Y;\varepsilon}$ are both local and inner regular

$0 \leq T_t^{M,Y} \varphi \leq T_t^M \varphi \leq T_t \varphi$ for all $\varphi \in L_2(X)_+$ and $t > 0$ by the last statement of Proposition 3.3 applied with S replaced by T . Combining these conclusions and substituting $T_t = S_t^{m,Y;\varepsilon}$ one deduces that

$$0 \leq S_{(1+\varepsilon)t}^{M,Y} \varphi \leq S_t^{m,Y;\varepsilon} \varphi \quad (15)$$

for all $\varphi \in L_2(X)_+$ and $t > 0$.

The forms $\mathcal{E}_{m,Y;\varepsilon}$ decrease monotonically as $\varepsilon \rightarrow 0$. Therefore it follows from Proposition 2.5 that the corresponding positive self-adjoint operators $H_{m,Y;\varepsilon}$ converge in the strong resolvent sense to the positive self-adjoint operator H_Y associated with the relaxation of the form $h_Y(\varphi) = \lim_{\varepsilon \rightarrow 0} \mathcal{E}_{m,Y;\varepsilon}(\varphi)$ on the domain $D(h_Y) = \bigcup_{\varepsilon > 0} D(\mathcal{E}_{m,Y;\varepsilon})$. But $D(h_Y) = D(\mathcal{E})$ and $h_Y(\varphi) = \mathcal{E}_{m,Y}(\varphi)$ for $\varphi \in D(\mathcal{E})$. Therefore H_Y is the operator corresponding to the relaxation $\mathcal{E}_{m,Y;0}$ of $\mathcal{E}_{m,Y}$ since $D(\mathcal{E})$ is a core of $\mathcal{E}_{m,Y}$ by Remark 3.2. Consequently $S_t^{m,Y;\varepsilon} \varphi \rightarrow S_t^{m,Y;0} \varphi$ as $\varepsilon \rightarrow 0$. Since $S_{(1+\varepsilon)t}^{M,Y} \varphi \rightarrow S_t^{M,Y} \varphi$ as $\varepsilon \rightarrow 0$ the assertion (14) follows immediately from (15). \square

Now we can establish the key order relation for S^M and S^m .

Proposition 3.6 *If \mathcal{E} is a local, inner regular, Dirichlet form and S^M, S^m are the submarkovian semigroups corresponding to the forms $\mathcal{E}_M, \mathcal{E}_m$ then*

$$0 \leq S_t^M \varphi \leq S_t^m \varphi \quad (16)$$

for all $\varphi \in L_2(X)_+$ and all $t > 0$. Consequently, $D(\mathcal{E}_M)$ is an order ideal of $D(\mathcal{E}_m)$ and $B(\mathcal{E}_M)$ is an algebraic ideal of $B(\mathcal{E}_m)$. Therefore $B_c(\mathcal{E}) = B_c(\mathcal{E}_M) = B_c(\mathcal{E}_m)$ and $C_c(\mathcal{E}) = C_c(\mathcal{E}_M) = C_c(\mathcal{E}_m)$.

Proof It follows from Proposition 3.3 that the net of semigroups $\{S^{M,Y}\}_{Y \in \mathcal{Y}}$ converges strongly to S^M . Next the generator of the semigroup S^m is the operator associated with the Dirichlet form \mathcal{E}_m defined by (9). Therefore \mathcal{E}_m is the supremum of the monotonically increasing net $\{\mathcal{E}_{m,Y;0}\}_{Y \in \mathcal{Y}}$ of Dirichlet forms $\mathcal{E}_{m,Y;0}$. Hence the corresponding semigroups $S^{m,Y;0}$ converge strongly to the semigroup S^m by Proposition 2.4. Then the semigroup ordering follows from (14). Explicitly one has

$$0 \leq S_t^M \varphi = \lim_{Y \in \mathcal{Y}} S^{M,Y} \varphi \leq \lim_{Y \in \mathcal{Y}} S^{m,Y;0} \varphi = S_t^m \varphi$$

for all $\varphi \in L_2(X)_+$ and all $t > 0$.

The assertions that $D(\mathcal{E}_M)$ is an order ideal of $D(\mathcal{E}_m)$ and $B(\mathcal{E}_M)$ is an algebraic ideal of $B(\mathcal{E}_m)$ now follow from Proposition 2.7 with \mathcal{E} replaced by \mathcal{E}_M , \mathcal{F} by \mathcal{E}_m , S_t by S_t^M and T_t by S_t^m and noting that $\mathcal{E}_m \supseteq \mathcal{E}_M$. Then Proposition 2.7 establishes that the order property $0 \leq S_t^M \varphi \leq S_t^m \varphi$ for all $\varphi \geq 0$ and $t > 0$ is equivalent to the ideal properties.

The last statement of the proposition follows because $B(\mathcal{E}_M)$ is an algebraic ideal of $B(\mathcal{E}_m)$. In particular $B_c(\mathcal{E}_M) B_c(\mathcal{E}_m) \subseteq B_c(\mathcal{E}_M) B(\mathcal{E}_m) \subseteq B_c(\mathcal{E}_M)$. Now if $\varphi \in B_c(\mathcal{E}_m)$ it follows from Lemma 2.1.II that one can choose $\eta \in C_c(\mathcal{E}_M)$ such that $0 \leq \eta \leq \mathbb{1}_X$ and $\eta = 1$ on the support of φ . Therefore $\eta \varphi = \varphi$. Hence $\varphi \in C_c(\mathcal{E}_M) B_c(\mathcal{E}_m) \subseteq B(\mathcal{E}_M)$. Thus $B_c(\mathcal{E}_m) \subseteq B_c(\mathcal{E}_M)$. But $\mathcal{E}_m \leq \mathcal{E} \leq \mathcal{E}_M$ and consequently $B_c(\mathcal{E}_m) \supseteq B_c(\mathcal{E}) \supseteq B_c(\mathcal{E}_M)$. Therefore $B_c(\mathcal{E}) = B_c(\mathcal{E}_M) = B_c(\mathcal{E}_m)$. Finally $C_c(\mathcal{E}) = B_c(\mathcal{E}) \cap C(X)$ etc. so one also deduces that $C_c(\mathcal{E}) = C_c(\mathcal{E}_M) = C_c(\mathcal{E}_m)$. \square

Proof of Theorem 3.1 continued Statements IV and V of Theorem 3.1 are established by the last statement of Proposition 3.6. It remains to prove Statement VI.

VI. First \mathcal{E}_M is regular and $C_c(\mathcal{E}_M) = C_c(\mathcal{E})$. Secondly, the inner regularity of \mathcal{E}_m follows from a slightly more general result.

Lemma 3.7 *Let \mathcal{E} be an inner regular Dirichlet form and \mathcal{F} a Dirichlet form extension of \mathcal{E}_M . Assume $B_c(\mathcal{F}) = B_c(\mathcal{E}_M)$.*

It follows that \mathcal{F} is inner regular and $\mathcal{F}_M = \mathcal{E}_M$.

Proof Since $B_c(\mathcal{F}) = B_c(\mathcal{E}_M)$ one also has $C_c(\mathcal{F}) = C_c(\mathcal{E}_M)$. But $\mathcal{E}_M = \overline{\mathcal{F}|_{C_c(\mathcal{E}_M)}}$ because \mathcal{F} is an extension of \mathcal{E}_M . Then

$$\mathcal{E}_M = \overline{\mathcal{F}|_{C_c(\mathcal{E}_M)}} = \overline{\mathcal{F}|_{C_c(\mathcal{F})}} \subseteq \overline{\mathcal{F}|_{B_c(\mathcal{F})}} = \overline{\mathcal{F}|_{B_c(\mathcal{E}_M)}} = \overline{\mathcal{E}_M|_{B_c(\mathcal{E}_M)}} = \overline{\mathcal{E}_M|_{C_c(\mathcal{E}_M)}} = \mathcal{E}_M$$

where the penultimate step uses the regularity of \mathcal{E}_M . Therefore $\overline{\mathcal{F}|_{C_c(\mathcal{F})}} = \overline{\mathcal{F}|_{B_c(\mathcal{F})}} = \mathcal{F}_M$ and \mathcal{F} is inner regular. Moreover, $\mathcal{F}_M = \overline{\mathcal{F}|_{C_c(\mathcal{F})}} = \mathcal{E}_M$. \square

The inner regularity of \mathcal{E}_m now follows from setting $\mathcal{F} = \mathcal{E}_m$ in Lemma 3.7 and noting that the assumptions of the lemma are satisfied by Statements III and IV of the theorem. This completes the proof of Theorem 3.1. \square

Since \mathcal{E}_M is a restriction of \mathcal{E} it follows that \mathcal{E}_M inherits the locality property from \mathcal{E} . It is, however, unclear if \mathcal{E}_m is local. This does follow if the forms $\mathcal{E}_{m,Y}$ are closable for all $Y \in \mathcal{Y}$ which in turn follows if the truncated forms \mathcal{E}_ξ are closable for all $\xi \in \mathcal{C}_Y(\mathcal{E})$ and $Y \in \mathcal{Y}$.

Proposition 3.8 *Let \mathcal{E} be a local (resp. strongly local), semi-regular, Dirichlet form. If the forms $\mathcal{E}_{m,Y}$ are closable for all $Y \in \mathcal{Y}$ then \mathcal{E}_m is local (resp. strongly local).*

Proof Since $\mathcal{E}_{m,Y}$ is closable the relaxation $\mathcal{E}_{m,Y;0}$ is automatically equal to its closure $\overline{\mathcal{E}_{m,Y}}$. Then \mathcal{E}_m is the monotonic limit of the increasing net $Y \in \mathcal{Y} \mapsto \overline{\mathcal{E}_{m,Y}}$. Thus

$$\mathcal{E}_m(\varphi) = \lim_{Y \in \mathcal{Y}} \overline{\mathcal{E}_{m,Y}}(\varphi) = \sup_{Y \in \mathcal{Y}} \overline{\mathcal{E}_{m,Y}}(\varphi) \quad (17)$$

for all $\varphi \in D(\mathcal{E}_m) = \{\varphi \in \bigcap_{Y \in \mathcal{Y}} D(\overline{\mathcal{E}_{m,Y}}) : \sup_{Y \in \mathcal{Y}} \overline{\mathcal{E}_{m,Y}}(\varphi) < \infty\}$.

Secondly, if \mathcal{E} is local the truncated functions \mathcal{E}_ξ are local on $B(\mathcal{E})$. Therefore $\mathcal{E}_\xi(|\varphi|) = \mathcal{E}_\xi(\varphi)$ for all $\xi \in \mathcal{C}_Y(\mathcal{E})$ and all $\varphi \in B(\mathcal{E})$. Hence

$$\mathcal{E}_{m,Y}(|\varphi|) = \lim_{\xi \in \mathcal{C}_Y(\mathcal{E})} \mathcal{E}_\xi(|\varphi|) = \lim_{\xi \in \mathcal{C}_Y(\mathcal{E})} \mathcal{E}_\xi(\varphi) = \mathcal{E}_{m,Y}(\varphi)$$

for all $\varphi \in B(\mathcal{E})$. It then follows from Proposition 2.3, applied with $\mathcal{E} = \overline{\mathcal{E}_{m,Y}}$ and $D = B(\mathcal{E})$, that $\overline{\mathcal{E}_{m,Y}}$ is local. Then \mathcal{E}_m is local by a similar argument.

The proof is analogous if \mathcal{E} is strongly local. The locality criterion $\mathcal{E}(|\varphi|) = \mathcal{E}(\varphi)$ of Proposition 2.3 is replaced by the corresponding criterion $\mathcal{E}(|\varphi + 1| - 1) = \mathcal{E}(\varphi)$ for strong locality. \square

Remark 3.9 The foregoing argument only requires locality, or strong locality, of the approximants $\{\mathcal{E}_{m,Y}\}_{Y \in \mathcal{Y}}$. This of course follows from locality, or strong locality, of \mathcal{E} although the latter property is not necessary. It is quite possible that \mathcal{E} is local but the $\mathcal{E}_{m,Y}$ are strongly local. In particular \mathcal{E}_m can be strongly local even if \mathcal{E} is only local (see the discussion of Robin boundary conditions at the end of Section 4).

The assumption that the forms $\mathcal{E}_{m,Y}$ are closable for all $Y \in \mathcal{Y}$ is not very satisfactory although it is satisfied for a large class of locally strongly elliptic operators. In fact the $\mathcal{E}_{m,Y}$ are often closed. For example, consider the Laplacian defined on the open subset Ω of \mathbf{R}^d with domain $C_c^\infty(\Omega)$. Then $\mathcal{E}_{m,Y}(\varphi) = \|\nabla \varphi\|_2^2$ with domain consisting of the $\varphi \in L_2(\Omega)$ whose restriction to Y is in $W^{1,2}(Y)$. It follows that $\mathcal{E}_{m,Y}$ is closable but if the boundary of Y is smooth, for example Lipschitz, then $\mathcal{E}_{m,Y}$ is closed.

Finally we give two examples of elliptic operators in one-dimension which illustrate the conclusions of Theorem 3.1. These examples are analyzed in detail in [RS10].

Example 3.10 Define the positive, symmetric, operator H on $L_2(\mathbf{R}_+)$ by $H\varphi = -(c\varphi')'$ where $c \in W_{\text{loc}}^{1,\infty}(\mathbf{R}_+)$ is strictly positive and $\varphi \in D(H) = C_c^\infty(\mathbf{R}_+)$. Let $\mathcal{E}_0(\varphi) = \int_0^\infty c(\varphi')^2$ with $D(\mathcal{E}_0) = C_c^\infty(\mathbf{R}_+)$. Set $\nu(x) = \int_x^1 c^{-1}$.

1. If $\nu \in L_\infty(0,1)$ then H has a one-parameter family of submarkovian extensions $H^{(\alpha)}$, where $\alpha \in [0, \infty]$, with corresponding Dirichlet forms $\mathcal{E}^{(\alpha)}$. If $\alpha \in [0, \infty)$ then $\mathcal{E}^{(\alpha)}(\varphi) = \mathcal{E}^{(0)}(\varphi) + \alpha |\varphi(0)|^2$ where $\mathcal{E}^{(0)}$ is the extension of \mathcal{E}_0 to the domain $D(\overline{\mathcal{E}_0}) + \text{span } \sigma_+$ with $\sigma_+ \in C_c^\infty(\mathbf{R}_+)$, $0 \leq \sigma_+ \leq 1$, $\sigma_+(x) = 1$ if $x \in \langle 0, 1 \rangle$ and $\sigma_+(x) = 0$ if $x \geq 2$ and where $D(\mathcal{E}^{(\alpha)}) = D(\mathcal{E}^{(0)})$. The associated operators satisfy the boundary conditions $(c\varphi')(0) = \alpha\varphi(0)$. In addition $\mathcal{E}^{(\infty)} = \overline{\mathcal{E}_0}$ and the corresponding boundary condition is $\varphi(0) = 0$. The family of forms $\mathcal{E}^{(\alpha)}$ is monotonically increasing with α and $\mathcal{E}^{(\infty)}$ formally corresponds to the limit $\alpha \rightarrow \infty$ of the $\mathcal{E}^{(\alpha)}$. These statements are contained in [RS10], Theorem 2.4.

The forms $\mathcal{E}^{(\alpha)}$ are all inner regular and $\mathcal{E}^{(\infty)}$ is regular. One can calculate explicitly the maximal and minimal forms $(\mathcal{E}^{(\alpha)})_M$ and $(\mathcal{E}^{(\alpha)})_m$. First it follows that $C_c(\mathcal{E}^{(\alpha)}) = C_c(\overline{\mathcal{E}_0})$ for all $\alpha \in [0, \infty]$. Therefore $(\mathcal{E}^{(\alpha)})_M = \overline{\mathcal{E}_0} = \mathcal{E}^{(\infty)}$ for all $\alpha \in [0, \infty]$. Secondly, $\mathcal{E}_\xi^{(\alpha)}(\varphi) = \int_Y \xi c(\varphi')^2$ for $\xi \in C_c(\mathcal{E}^{(\alpha)})$ with $\text{supp } \xi \subseteq Y$ and $\varphi \in B(\mathcal{E}^{(\alpha)})$. Hence $\mathcal{E}_{m,Y}^{(\alpha)}(\varphi) = \int_Y c(\varphi')^2$ for all $\varphi \in B(\mathcal{E}^{(\alpha)})$. But if $\alpha \in [0, \infty)$ then $B(\mathcal{E}^{(\alpha)}) = B(\mathcal{E}^{(0)})$. Alternatively $\mathcal{E}_{m,Y}^{(\infty)}$ extends to $B(\mathcal{E}^{(0)})$ by continuity. Then by definition $(\mathcal{E}^{(\alpha)})_m = \mathcal{E}^{(0)}$ for all $\alpha \in [0, \infty]$. Thus the maximal and minimal forms are independent of the choice of α and are identified with the maximal and minimal forms in the family $\mathcal{E}^{(\alpha)}$. The minimal form $\mathcal{E}^{(0)}$ is an extension of the maximal form $\mathcal{E}^{(\infty)}$ but not of the intermediate forms $\mathcal{E}^{(\alpha)}$ with $\alpha \in \langle 0, \infty \rangle$.

2. If $\nu \notin L_\infty(0,1)$ then H has a unique submarkovian extension determined by the form $\overline{\mathcal{E}_0}$. The associated self-adjoint operator satisfies the boundary condition $(c\varphi')(0) = 0$. In this case $\mathcal{E}_M = \mathcal{E}_m = \overline{\mathcal{E}_0}$.

The diffusion process described by the first case of the example is determined by its behaviour at the boundary point, i.e. at the origin. This behaviour is dependent on the boundary value of the diffusion coefficient c . In the second case the coefficient is sufficiently degenerate at the origin that the diffusion fails to reach the boundary. Therefore there is no ambiguity and the diffusion is uniquely determined. A different phenomenon can occur if the diffusion is degenerate at an interior point. This is illustrated by the second example which also clarifies the significance of inner regularity.

Example 3.11 Define the positive, symmetric, operator H on $L_2(\mathbf{R})$ by $H\varphi = -(c\varphi')'$ where $\varphi \in D(H) = C_c^\infty(\mathbf{R})$ and $c \in W_{\text{loc}}^{1,\infty}(\mathbf{R})$ is strictly positive on $\mathbf{R} \setminus \{0\}$ but $c(0) = 0$. Let $\mathcal{E}_0(\varphi) = \int_{-\infty}^\infty c(\varphi')^2$ with $D(\mathcal{E}_0) = C_c^\infty(\mathbf{R})$. Set $\nu_+(x) = \int_x^1 c^{-1}$ and $\nu_- = \int_{-1}^{-x} c^{-1}$.

1. If $\nu_+ \vee \nu_- \in L_\infty(0, 1)$ then H has a one-parameter family of submarkovian extensions $H^{(\alpha)}$, where $\alpha \in [0, \infty]$. If $\alpha \in [0, \infty)$ then these extensions are determined by the Dirichlet forms

$$\mathcal{E}^{(\alpha)}(\varphi) = \mathcal{E}^{(0)}(\varphi) + \alpha |\varphi(0_+) - \varphi(0_-)|^2$$

where $\mathcal{E}^{(0)}$ is the extension of \mathcal{E}_0 to the domain $D(\overline{\mathcal{E}_0}) + \text{span}(\sigma_+ - \sigma_-)$ and $D(\mathcal{E}^{(\alpha)}) = D(\mathcal{E}^{(0)})$. Here σ_+ is the function on \mathbf{R}_+ defined in Example 3.10 and σ_- is defined on \mathbf{R}_- by $\sigma_-(x) = \sigma_+(-x)$. The domain of the corresponding self-adjoint operator is characterized by the continuity condition $(c\varphi')(0_+) - (c\varphi')(0_-) = \alpha(\varphi(0_+) - \varphi(0_-))$. In addition $\mathcal{E}^{(\infty)} = \overline{\mathcal{E}_0}$ which again corresponds to the limit $\alpha \rightarrow \infty$. (These statements are established in [RS10], Theorem 1.1.) One can again calculate the maximal and minimal forms.

First $C_c(\mathcal{E}^{(\alpha)}) = C_c(\overline{\mathcal{E}_0})$ for all $\alpha \in [0, \infty]$. (Note that here it is essential that we are considering the subspace of $D(\mathcal{E}^{(\alpha)})$ formed by continuous functions with compact support. A similar identification is not true for the subspaces $B_c(\mathcal{E}^{(\alpha)})$ of bounded functions unless $\alpha = \infty$.) Therefore $(\mathcal{E}^{(\alpha)})_M = \overline{\mathcal{E}_0} = \mathcal{E}^{(\infty)}$ for all $\alpha \in [0, \infty]$.

Secondly, the identification of the minimal forms has two distinct cases. In the limit case $\alpha = \infty$ one has $\mathcal{E}^{(\infty)} = \overline{\mathcal{E}_0}$ and $\mathcal{E}^{(\infty)}$ is regular. Then $\mathcal{E}_{m,Y}^{(\infty)}(\varphi) = \int_Y c(\varphi')^2$ for all $\varphi \in D(\overline{\mathcal{E}_0})$ and each bounded interval Y . Therefore $(\mathcal{E}^{(\infty)})_m = \overline{\mathcal{E}_0} = (\mathcal{E}^{(\infty)})_M$ and the statements of Theorem 3.1 are obviously valid.

Thirdly, if $\alpha \in [0, \infty)$ then $C_c(\mathcal{E}^{(\alpha)}) = C_c(\overline{\mathcal{E}_0})$ but $B_c(\mathcal{E}^{(\alpha)}) = B_c(\overline{\mathcal{E}_0}) + \text{span}(\sigma_+ - \sigma_-) = C_c(\overline{\mathcal{E}_0}) + \text{span}(\sigma_+ - \sigma_-)$. Therefore the $\mathcal{E}^{(\alpha)}$ are not inner regular and the conclusions of Theorem 3.1 do not apply. In particular $B_c(\mathcal{E}^{(\alpha)}) \neq B_c(\overline{\mathcal{E}_0})$. In fact the forms $(\mathcal{E}^{(\alpha)})_m$ are not equal to the minimal form $\mathcal{E}^{(0)}$ of the family $\mathcal{E}^{(\alpha)}$. Instead one has $(\mathcal{E}^{(\alpha)})_m = \mathcal{E}^{(\alpha)}$.

2. If $\nu_+ \vee \nu_- \notin L_\infty(0, 1)$ then H has a unique submarkovian extension determined by the Dirichlet form $\mathcal{E} = \overline{\mathcal{E}_0}$. Its domain is characterized by the continuity conditions $(c\varphi')(0_+) = 0 = (c\varphi')(0_-)$. In this case $\mathcal{E} = \mathcal{E}_M = \mathcal{E}_m$. The semigroup corresponding to the unique Dirichlet form \mathcal{E} leaves the subspaces $L_2(\mathbf{R}_\pm)$ invariant. The diffusion coefficient c is sufficiently degenerate that there is no transmission past the origin which acts as a virtual boundary. (Again see [RS10], Theorem 1.1.)

The foregoing example demonstrates that interior regularity properties are related to the interior degeneracy structure. Similar effects occur in higher dimensions. Local degeneracies lead to local discontinuities.

4 Local extensions

In this section we examine local Dirichlet forms \mathcal{F} which extend the local, inner regular, Dirichlet form \mathcal{E} or, more generally, extend \mathcal{E}_M . Our aim is to characterize the forms with the property $\mathcal{E}_m \leq \mathcal{F} \leq \mathcal{E}_M$. Since \mathcal{F} extends \mathcal{E}_M it is automatically semi-regular and $\mathcal{F} \leq \mathcal{E}_M$. The semi-regularity suffices to define the extremal forms $\mathcal{F}_m, \mathcal{F}_M$. Then $\mathcal{F}_m \leq \mathcal{E}_m$ and $\mathcal{F}_M \leq \mathcal{E}_M$. Thus the lower bound $\mathcal{E}_m \leq \mathcal{F}$ and its optimality depend on the equality $\mathcal{E}_m = \mathcal{F}_m$. Similarly the optimality of the upper bound $\mathcal{F} \leq \mathcal{E}_M$ requires $\mathcal{F}_M = \mathcal{E}_M$. In fact the optimality is related to the inner regularity.

Theorem 4.1 *Let \mathcal{E} be a local, inner regular, Dirichlet form and \mathcal{F} a local Dirichlet form extension of \mathcal{E}_M .*

The following conditions are equivalent:

- I. $\mathcal{E}_m \leq \mathcal{F} \leq \mathcal{E}_M$,
- II. $B_c(\mathcal{E}) = B_c(\mathcal{F})$.

Moreover, if these conditions are satisfied then \mathcal{F} is inner regular, $\mathcal{F}_M = \mathcal{E}_M$ and $\mathcal{F}_m = \mathcal{E}_m$.

Proof I \Rightarrow II. It follows from the order relation that $B_c(\mathcal{E}_m) \supseteq B_c(\mathcal{F}) \supseteq B_c(\mathcal{E}_M)$. But $B_c(\mathcal{E}_m) = B_c(\mathcal{E}) = B_c(\mathcal{E}_M)$ by Theorem 3.1. Therefore $B_c(\mathcal{E}) = B_c(\mathcal{F})$.

II \Rightarrow I. First it follows from Condition II and Lemma 3.7 that \mathcal{F} is inner regular. But $B_c(\mathcal{F}) = B_c(\mathcal{E})$ and $B_c(\mathcal{E}) = B_c(\mathcal{E}_M)$ by Theorem 3.1. Therefore $B_c(\mathcal{F}) = B_c(\mathcal{E}_M)$. Consequently $C_c(\mathcal{F}) = C_c(\mathcal{E}_M)$. Since $C_c(\mathcal{F}) = C_c(\mathcal{F}_M)$ it follows that $C_c(\mathcal{E}_M) = C_c(\mathcal{F}_M)$. Therefore $\mathcal{F}_M = \mathcal{E}_M$. Then, however, $\mathcal{E}_m = (\mathcal{E}_M)_m = (\mathcal{F}_M)_m = \mathcal{F}_m$ by Theorem 3.1 applied first to \mathcal{E} and then to \mathcal{F} . The latter application is valid since \mathcal{F} is inner regular. One then has $\mathcal{E}_m = \mathcal{F}_m \leq \mathcal{F} \leq \mathcal{F}_M = \mathcal{E}_M$.

The last statement of the theorem has been established by the foregoing argument. \square

If one assumes that the form \mathcal{F} is inner regular then it follows by the foregoing argument that $\mathcal{E}_M = \mathcal{F}_M$ implies $\mathcal{E}_m = \mathcal{F}_m$. Conversely, if $\mathcal{E}_m = \mathcal{F}_m$ then $\mathcal{E}_M = (\mathcal{E}_m)_M = (\mathcal{F}_m)_M = \mathcal{F}_M$. But then the conditions $\mathcal{E}_M = \mathcal{F}_M$ and $\mathcal{E}_m = \mathcal{F}_m$ imply that $\mathcal{E}_m \leq \mathcal{F} \leq \mathcal{E}_M$. Therefore one has the following conclusion.

Corollary 4.2 *If \mathcal{E} and \mathcal{F} are local, inner regular, Dirichlet forms with $\mathcal{F} \supseteq \mathcal{E}_M$ then the following conditions are equivalent:*

- I. $\mathcal{E}_m \leq \mathcal{F} \leq \mathcal{E}_M$,
- II. $\mathcal{F}_M = \mathcal{E}_M$,
- III. $\mathcal{F}_m = \mathcal{E}_m$.

In fact much more is true under the stronger regularity assumption.

Corollary 4.3 *Assume \mathcal{E} and \mathcal{F} are local, inner regular, Dirichlet forms with $\mathcal{F} \supseteq \mathcal{E}_M$. Let S^M and T denote the submarkovian semigroups associated with \mathcal{E}_M and \mathcal{F} , respectively. Then the following conditions are equivalent:*

- I. $\mathcal{E}_m \leq \mathcal{F} \leq \mathcal{E}_M$,
- II. $0 \leq S_t^M \varphi \leq T_t \varphi$ for all $\varphi \in L_2(X)_+$ and all $t > 0$,
- III. $D(\mathcal{E}_M)$ is an order ideal of $D(\mathcal{F})$,
- IV. $B(\mathcal{E}_M)$ is an algebraic ideal of $B(\mathcal{F})$.

Proof The mutual equivalence of the last three conditions follows from Proposition 2.7 with \mathcal{E} replaced by \mathcal{E}_M . Since $\mathcal{F} \supseteq \mathcal{E}_M$ the off-diagonal bounds in Conditions II and III of Proposition 2.7 are automatically fulfilled. Therefore these conditions reduce to the order ideal property and the algebraic ideal property, respectively. The equivalences of Proposition 2.7 are independent of any locality or regularity assumptions. It remains to prove equivalence of the last three conditions with the first condition.

I \Rightarrow II. First Condition I is equivalent to the condition $\mathcal{E}_M = \mathcal{F}_M$ by Corollary 4.2. In particular $S^M = T^M$ where T^M denotes the submarkovian semigroup associated with the Dirichlet form \mathcal{F}_M . But then $0 \leq S_t^M \varphi = T_t^M \varphi \leq T_t \varphi$ for all $\varphi \in L_2(X)_+$ and all $t > 0$

by the last statement of Proposition 3.3 applied to \mathcal{F} . Here it is important that \mathcal{F} is inner regular.

III \Rightarrow I. Condition I is equivalent to the condition $B_c(\mathcal{E}_M) = B_c(\mathcal{E}) = B_c(\mathcal{F})$ by Theorem 4.1. Now we argue by negation.

Assume $B_c(\mathcal{E}_M) \neq B_c(\mathcal{F})$. Since $B_c(\mathcal{F}) \supseteq B_c(\mathcal{E}_M)$ it follows that there is a non-zero $\psi \in B_c(\mathcal{F})$ such that $\psi \notin B_c(\mathcal{E}_M)$. Replacing ψ by $|\psi|$ and rescaling if necessary we may assume $0 \leq \psi \leq 1$. Since \mathcal{E} is semi-regular one may then choose $\varphi \in C_c(\mathcal{E}_M)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $\text{supp } \psi$ by Lemma 2.1. Therefore $0 \leq \psi \leq \varphi$ and since $\psi \notin B_c(\mathcal{E}_M)$ this contradicts the order ideal property. \square

Note that Theorem 4.1 incorporates the case of local Dirichlet forms \mathcal{F} which satisfy $\mathcal{E}_M \subseteq \mathcal{F} \subseteq \mathcal{E}_m$. This extension/restriction condition immediately implies the ordering condition $\mathcal{E}_M \geq \mathcal{F} \geq \mathcal{E}_m$. Consequently the conclusions of the theorem are valid.

Example 3.10 demonstrates that the conclusions of Theorems 4.1 are essentially optimal. A similar situation occurs in higher dimensions but then there is a much wider range of possibilities and a greater variety of strongly local extensions.

Example 4.4 Let Ω be a domain, i.e. an open connected set, in \mathbf{R}^d and \mathcal{E}_0 the Markovian form with $D(\mathcal{E}_0) = C_c^\infty(\Omega)$ given by

$$\mathcal{E}_0(\varphi) = \sum_{k,l=1}^d (\partial_k \varphi, c_{kl} \partial_l \varphi)$$

where $c_{kl} = c_{lk} \in L_{\infty, \text{loc}}(\Omega)$ and $C = (c_{kl})$ is locally strongly elliptic, i.e. for each compact subset $K \subseteq \Omega$ there is a $c_K > 0$ such that $C(x) \geq c_K I$ for (almost) all $x \in \Omega$. Then \mathcal{E}_0 is closable (see [MR92] Section II.2.b). Since the truncation $\mathcal{E}_{0,\xi}$ of \mathcal{E}_0 corresponds to the replacement $c_{kl} \mapsto \xi c_{kl}$ the truncated forms with $\xi \geq 0$ are also closable.

Let $\mathcal{E}_M = \overline{\mathcal{E}_0}$. Then $\mathcal{E}_m (= (\mathcal{E}_M)_m)$ is given by

$$\mathcal{E}_m(\varphi) = \int_{\Omega} dx \Gamma(\varphi)(x), \quad (18)$$

where $\Gamma(\varphi) = \sum_{k,l=1}^d c_{kl} (\partial_k \varphi) (\partial_l \varphi)$, with $D(\mathcal{E}_m) = \{\varphi \in W_{\text{loc}}^{1,2}(\Omega) : \Gamma(\varphi) + \varphi^2 \in L_1(\Omega)\}$. The forms \mathcal{E}_m and \mathcal{E}_M are the extremal forms for each Dirichlet form extension \mathcal{E} of \mathcal{E}_0 , i.e. $\mathcal{E}_m \leq \mathcal{E} \leq \mathcal{E}_M$. The form \mathcal{E}_M is regular, \mathcal{E}_m is inner regular and both forms are strongly local.

If $c_{kl} \in W_{\text{loc}}^{1,\infty}(\Omega)$ then \mathcal{E}_0 is the form of the operator $H_0 = -\sum_{k,l=1}^d \partial_k c_{kl} \partial_l$ with $D(H_0) = C_c^\infty(\Omega)$. Then H_M , the operator corresponding to \mathcal{E}_M , is the submarkovian extension of H_0 satisfying Dirichlet conditions $\varphi|_{\partial\Omega} = 0$ on the boundary $\partial\Omega$. The operator H_m corresponding to \mathcal{E}_m formally satisfies Neumann conditions, i.e. if $\partial\Omega$ is Lipschitz then $(n \cdot C \nabla \varphi)|_{\partial\Omega} = 0$. The forms \mathcal{E}_F defined as the closure of \mathcal{E}_m restricted to $\{\varphi|_{\Omega} : \varphi \in C_c^\infty(\mathbf{R}^d \setminus F)\}$, where F is a closed subset of $\partial\Omega$, are all inner regular, strongly local, Dirichlet form extensions of \mathcal{E}_0 which are bounded above and below by \mathcal{E}_M and \mathcal{E}_m , respectively. The corresponding operators H_F satisfy a mixture of conditions, Dirichlet on F and Neumann on $\partial\Omega \setminus F$. The semigroups S^F generated by the H_F dominate the semigroup S^M generated by H_M . In fact since $\mathcal{E}_F \subseteq \mathcal{E}_m$ and $D(\mathcal{E}_F)$ is an order ideal of $D(\mathcal{E}_m)$ the semigroup S^m generated by H_m dominates the S^F . Thus $0 \leq S_t^M \varphi \leq S_t^F \varphi \leq S_t^m \varphi$ for all $\varphi \geq 0$, all $t > 0$ and all choices of F .

There are also Dirichlet form extensions corresponding to Robin boundary conditions. If one assumes, for simplicity, that $\partial\Omega$ is Lipschitz and sets

$$\mathcal{E}^{(\alpha)}(\varphi) = \mathcal{E}_m(\varphi) + \int_{\partial\Omega} dS \alpha |\varphi|^2 \quad (19)$$

where dS is the surface measure, α a positive bounded function on $\partial\Omega$ and $D(\mathcal{E}^{(\alpha)})$ consists of those $\varphi \in D(\mathcal{E}_m)$ then the operators H_α corresponding to the $\mathcal{E}^{(\alpha)}$ satisfy the Robin boundary conditions $(n \cdot C \nabla \varphi + \alpha \varphi)|_{\partial\Omega} = 0$. One has $\mathcal{E}_m \leq \mathcal{E}^{(\alpha)} \leq \mathcal{E}_M$ and $\mathcal{E}^{(\alpha)} \supseteq \mathcal{E}_M$ but \mathcal{E}_m is not an extension of $\mathcal{E}^{(\alpha)}$. Nevertheless \mathcal{E}_m and \mathcal{E}_M are the extremal forms for the $\mathcal{E}^{(\alpha)}$. The construction of \mathcal{E}_M and \mathcal{E}_m is by ‘interior’ approximation and the α -boundary term plays no role. The $\mathcal{E}^{(\alpha)}$ with $\alpha \neq 0$ are local, inner regular, forms but they are not strongly local, because of the boundary term. One again has the domination properties $0 \leq S_t^M \varphi \leq S_t^{(\alpha)} \varphi \leq S_t^m \varphi$ for all $\varphi \geq 0$, all $t > 0$ and all choices of $\alpha \geq 0$. The order relation between S^M and $S^{(\alpha)}$ follows from the last statement of Proposition 3.3 applied with S replaced by $S^{(\alpha)}$. The order relation between $S^{(\alpha)}$ and S^m follows from Proposition 2.7 applied to $\mathcal{E}^{(\alpha)}$ and \mathcal{E}_m . Since $D(\mathcal{E}^{(\alpha)}) = D(\mathcal{E}_m)$ the ideal properties of Conditions II and III of Proposition 2.7 are evident. But the off-diagonal bound $\mathcal{E}^{(\alpha)}(\varphi, \psi) \geq \mathcal{E}_m(\varphi, \psi)$ for $\varphi, \psi \in D(\mathcal{E}_m)_+$ follows because $\alpha \geq 0$. These features also extend to the broader class of Robin conditions given in [Dan00] [AW03b]. \square

This example illustrates that the locality properties are quite complex. If one chooses the strongly local Dirichlet form \mathcal{E}_M as starting point then the minimal form $\mathcal{E}_m = (\mathcal{E}_M)_m$ is strongly local, as are the intermediate forms \mathcal{E}_F , but the Robin forms $\mathcal{E}^{(\alpha)}$ are only local. If, however, one chooses the local Robin form $\mathcal{E}^{(\alpha)}$ as starting point then the extremal forms are still given by \mathcal{E}_m and \mathcal{E}_M and they are both strongly local. Thus in the latter case there is a strengthening of the locality property.

If in Example 4.4 the coefficients $c_{kl} \in W_{\text{loc}}^{1,\infty}(\Omega)$ then the various self-adjoint operators H_m, H_M, H_F and $H^{(\alpha)}$ associated with the forms $\mathcal{E}_m, \mathcal{E}_M, \mathcal{E}_F$ and $\mathcal{E}^{(\alpha)}$ are all submarkovian extensions of the symmetric operator H_0 . In particular H_m is the smallest such extension and H_M the largest, e.g. $H_m \leq H_F \leq H_M$ and $H_m \leq H^{(\alpha)} \leq H_M$. Therefore H_0 is Markov unique, i.e. it has a unique submarkovian extension, if and only if $H_m = H_M$ or, equivalently, $\mathcal{E}_m = \mathcal{E}_M$. In the next section we examine this latter uniqueness criterion in the general situation.

5 Applications

In this section we consider several implications of the foregoing results. We concentrate on two topics, the dependence on \mathcal{E} of the set-theoretic distance function $d^{(\mathcal{E})}(\cdot; \cdot)$ canonically associated with the form and the uniqueness criterion $\mathcal{E}_m = \mathcal{E}_M$. In contrast to the earlier discussion strong locality is now essential for much of the analysis. In fact strong locality of the form \mathcal{E}_m is of prime importance. This implies strong locality of \mathcal{E}_M , because $\mathcal{E}_m \supseteq \mathcal{E}_M$, but it does not necessarily require strong locality of \mathcal{E} (see Remark 3.9 and the discussion following Example 4.4). The second key ingredient in the discussion is the algebraic ideal property, i.e. the observation that $B(\mathcal{E}_M)$ is an algebraic ideal of $B(\mathcal{E}_m)$.

5.1 Distances

One can associate with a general strongly local Dirichlet form \mathcal{E} a set-theoretic distance function [Stu98] [HR03] [AH05] [ERSZ06]. In particular the distance as defined by Ariyoshi and Hino [AH05] for a strongly local Dirichlet form \mathcal{E} on the measure space X, μ is independent of any topology. It is a positive function $d^{(\mathcal{E})}(A; B)$ over pairs of measurable sets A, B with $\mu(A), \mu(B) \in \langle 0, \infty \rangle$. We next sketch the construction of this function and refer to [AH05] for details.

First for each closed subset A of X set

$$D_A(\mathcal{E}) = \{\varphi \in D(\mathcal{E}) : \text{supp } \varphi = A\}$$

and $B_A(\mathcal{E}) = D_A(\mathcal{E}) \cap L_\infty(X)$. Secondly, following [AH05] Definition 2.1, define a nest as an increasing family $\mathcal{A} = \{A_\lambda\}_{\lambda>0}$ of sets A_λ with finite \mathcal{E} -capacity such that $D_{\mathcal{A}}(\mathcal{E}) = \bigcup_{\lambda>0} D_{A_\lambda}(\mathcal{E})$ is a core of \mathcal{E} or, equivalently, that $B_{\mathcal{A}}(\mathcal{E}) = \bigcup_{\lambda>0} B_{A_\lambda}(\mathcal{E})$ is a core. It is not evident that nests of this type exist but this follows from [AH05] Lemma 3.1 by an argument based on the theory of excessive functions.

Thirdly define the local subspace of $D(\mathcal{E})$ corresponding to the nest \mathcal{A} by

$$D_{\mathcal{A};\text{loc}}(\mathcal{E}) = \{\varphi \in M(X) : \text{there exist } \varphi_\lambda \in D(\mathcal{E}) \text{ such that } \varphi = \varphi_\lambda \text{ on } A_\lambda\}$$

and $B_{\mathcal{A};\text{loc}}(\mathcal{E}) = D_{\mathcal{A};\text{loc}} \cap L_\infty(X)$ (see [AH05] Definition 2.4). Since the A_λ are sets of finite \mathcal{E} -capacity it immediately follows that $\mathbf{1} \in B_{\mathcal{A};\text{loc}}(\mathcal{E})$. Moreover, if F is a normal contraction and $\varphi \in D_{\mathcal{A};\text{loc}}(\mathcal{E})$ then $F \circ \varphi \in D_{\mathcal{A};\text{loc}}(\mathcal{E})$ since \mathcal{E} is a Dirichlet form.

Fourthly if \mathcal{E} is local one can extend the earlier definition of the truncated forms to the local functions. It is convenient for the sequel to adopt the notation of [ERSZ06] and set

$$\mathcal{I}_\varphi^{(\mathcal{E})}(\xi) = \mathcal{E}_\xi(\varphi) = \mathcal{E}(\varphi, \xi \varphi) - 2^{-1} \mathcal{E}(\xi, \varphi^2)$$

for $\xi \in B(\mathcal{E})_+$ and $\varphi \in D(\mathcal{E}_\xi) = B(\mathcal{E})$. It then follows from the locality of \mathcal{E} that $\mathcal{I}_\varphi^{(\mathcal{E})}(\xi) = 0$ if $\varphi \xi = 0$. Consequently for each $\varphi \in B_{\mathcal{A};\text{loc}}(\mathcal{E})$ one can introduce the form $\hat{\mathcal{I}}_\varphi^{(\mathcal{E})}$ on $B_{\mathcal{A}}(\mathcal{E})_+$ by setting

$$\hat{\mathcal{I}}_\varphi^{(\mathcal{E})}(\xi) = \mathcal{I}_{\varphi_\lambda}^{(\mathcal{E})}(\xi)$$

for $\xi \in B_{A_\lambda}(\mathcal{E})_+$ and for any $\varphi_\lambda \in D(\mathcal{E})$ with $\varphi|_{A_\lambda} = \varphi_\lambda$.

Finally define

$$||| \hat{\mathcal{I}}_\varphi^{(\mathcal{E})} ||| = \sup\{\hat{\mathcal{I}}_\varphi^{(\mathcal{E})}(\xi) : \xi \in B_{\mathcal{A}}(\mathcal{E})_+, \|\xi\|_1 \leq 1\}$$

and introduce

$$D_{0,\mathcal{A}}(\mathcal{E}) = \{\varphi \in B_{\mathcal{A};\text{loc}}(\mathcal{E}) : ||| \hat{\mathcal{I}}_\varphi^{(\mathcal{E})} ||| \leq 1\}.$$

This corresponds to Definition 2.6 of [AH05]. It then follows from Proposition 3.9 of [AH05] that if \mathcal{E} is strongly local then $D_{0,\mathcal{A}}(\mathcal{E})$ is independent of the particular choice of nest, i.e. $D_{0,\mathcal{A}}(\mathcal{E}) = D_{0,\mathcal{B}}(\mathcal{E})$ for any pair of nests \mathcal{A}, \mathcal{B} . Hence one may simplify notation by setting $D_0(\mathcal{E}) = D_{0,\mathcal{A}}(\mathcal{E})$. Then for each pair of measurable sets $A, B \subset X$ with $\mu(A), \mu(B) > 0$ the distance between A and B corresponding to \mathcal{E} is defined by

$$d^{(\mathcal{E})}(A; B) = \sup \left\{ \text{ess inf}_{x \in A} \varphi(x) - \text{ess sup}_{y \in B} \varphi(y) : \varphi \in D_0(\mathcal{E}) \right\}.$$

This definition agrees with that of Ariyoshi and Hino and we use their results in the sequel.

At this point we return to the setting of Sections 3 and 4 and compare the distances corresponding to extensions of a strongly local inner regular form \mathcal{E} . The algebraic ideal properties are of importance. Recall that if \mathcal{E} and \mathcal{F} are two Dirichlet forms with $\mathcal{F} \geq \mathcal{E}$ then $D(\mathcal{F}) \subseteq D(\mathcal{E})$. Therefore $B_{\mathcal{A}}(\mathcal{F}) \subseteq B_{\mathcal{A}}(\mathcal{E})$ and $B_{\mathcal{A};\text{loc}}(\mathcal{F}) \subseteq B_{\mathcal{A};\text{loc}}(\mathcal{E})$. Further if $B(\mathcal{F})$ is an algebraic ideal of $B(\mathcal{E})$, i.e. if $B(\mathcal{F})B(\mathcal{E}) \subseteq B(\mathcal{F})$, then $B_{\mathcal{A};\text{loc}}(\mathcal{F})B_{\mathcal{A}}(\mathcal{E}) \subseteq B_{\mathcal{A};\text{loc}}(\mathcal{F})$.

The principal result concerns the extremal forms.

Theorem 5.1 *Let \mathcal{E} be an inner regular, strongly local, Dirichlet form and \mathcal{E}_m , \mathcal{E}_M the corresponding minimal and maximal Dirichlet forms. Then*

$$d^{(\mathcal{E})}(A; B) = d^{(\mathcal{E}_M)}(A; B)$$

for all measurable subsets A, B with $\mu(A), \mu(B) \in \langle 0, \infty \rangle$. Moreover, if \mathcal{E}_m is strongly local then one also has

$$d^{(\mathcal{E}_m)}(A; B) = d^{(\mathcal{E})}(A; B)$$

for all A, B with $\mu(A), \mu(B) \in \langle 0, \infty \rangle$.

Proof First strong locality of \mathcal{E}_M follows from strong locality of \mathcal{E} since $\mathcal{E}_M \subseteq \mathcal{E}$. Therefore $d^{(\mathcal{E})}$ and $d^{(\mathcal{E}_M)}$ are both well-defined. Now we begin by proving that $d^{(\mathcal{E}_M)}(A; B) \leq d^{(\mathcal{E})}(A; B)$ for all A, B . The proof is a variation of the argument used to establish Proposition 5.3 in [ERSZ06].

Fix $\varphi \in B_{\mathcal{A};\text{loc}}(\mathcal{E}_M)_+$ with $|||\hat{\mathcal{I}}_\varphi^{(\mathcal{E}_M)}||| < \infty$. Now $\mathbf{1} \in B_{\mathcal{A};\text{loc}}(\mathcal{E}_M)_+$ and if $\xi \in B_{\mathcal{A}}(\mathcal{E})_+$ then

$$\begin{aligned} \hat{\mathcal{I}}_{\mathbf{1}+\varphi}^{(\mathcal{E})}(\xi) &= \mathcal{E}((\varphi_\lambda + \mathbf{1}_\lambda), \xi(\varphi_\lambda + \mathbf{1}_\lambda)) - 2^{-1}\mathcal{E}(\xi, (\varphi_\lambda + \mathbf{1}_\lambda)^2) \\ &= \mathcal{E}(\varphi_\lambda, \xi\varphi_\lambda) + \mathcal{E}(\varphi_\lambda, \xi) - 2^{-1}(\mathcal{E}(\xi, \varphi_\lambda^2) + 2\mathcal{E}(\xi, \varphi_\lambda)) = \hat{\mathcal{I}}_\varphi^{(\mathcal{E})}(\xi) \end{aligned}$$

by strong locality of \mathcal{E} . Therefore $\hat{\mathcal{I}}_\varphi^{(\mathcal{E})}(\xi) = \hat{\mathcal{I}}_{\mathbf{1}+\varphi}^{(\mathcal{E})}(\xi)$ for all $\xi \in B_{\mathcal{A}}(\mathcal{E})_+$.

Next $(\mathbf{1} + \varphi)^{1/2} = \mathbf{1} + F \circ \varphi$ with $F(x) = (1 + |x|)^{1/2} - 1$. But F is a normal contraction. Therefore $(\mathbf{1} + \varphi)^{1/2} \in B_{\mathcal{A};\text{loc}}(\mathcal{E}_M)_+$. Now the polarized form

$$\mathcal{I}_{\varphi, \psi}^{(\mathcal{E})}(\xi) = \mathcal{E}_\xi(\varphi, \psi) = 2^{-1} \left(\mathcal{E}(\varphi, \xi\psi) + \mathcal{E}(\xi\varphi, \psi) - \mathcal{E}(\xi, \varphi\psi) \right)$$

satisfies the Leibniz relation

$$\mathcal{I}_{\varphi_1\varphi_2, \psi}^{(\mathcal{E})}(\xi) = \mathcal{I}_{\varphi_1, \psi}^{(\mathcal{E})}(\varphi_2\xi) + \mathcal{I}_{\varphi_2, \psi}^{(\mathcal{E})}(\varphi_1\xi)$$

as a consequence of strong locality. It then readily follows that

$$\hat{\mathcal{I}}_\varphi^{(\mathcal{E})}(\xi) = \hat{\mathcal{I}}_{\mathbf{1}+\varphi}^{(\mathcal{E})}(\xi) = 4\hat{\mathcal{I}}_{(\mathbf{1}+\varphi)^{1/2}}^{(\mathcal{E})}((\mathbf{1} + \varphi)\xi) \quad (20)$$

for all $\xi \in B_{\mathcal{A}}(\mathcal{E})_+$. But $\eta = (\mathbf{1} + \varphi)\xi \in B_{\mathcal{A};\text{loc}}(\mathcal{E}_M)_+$ by the ideal property. Moreover, $\hat{\mathcal{I}}_{(\mathbf{1}+\varphi)^{1/2}}^{(\mathcal{E})}(\eta) \leq \hat{\mathcal{I}}_{(\mathbf{1}+\varphi)^{1/2}}^{(\mathcal{E}_M)}(\eta)$ for all $\eta \in B_{\mathcal{A}}(\mathcal{E}_M)_+$ by the order property $\mathcal{E} \leq \mathcal{E}_M$. Therefore

$$\hat{\mathcal{I}}_\varphi^{(\mathcal{E})}(\xi) \leq 4\hat{\mathcal{I}}_{(\mathbf{1}+\varphi)^{1/2}}^{(\mathcal{E}_M)}((\mathbf{1} + \varphi)\xi)$$

for all $\xi \in B_{\mathcal{A}}(\mathcal{E})_+$. It immediately follows that

$$||| \hat{\mathcal{I}}_{\varphi}^{(\mathcal{E})} ||| \leq 4(1 + \|\varphi\|_{\infty}) ||| \hat{\mathcal{I}}_{(1+\varphi)^{1/2}}^{(\mathcal{E}_M)} |||.$$

Then, however, one deduces from (20) with \mathcal{E} replaced by \mathcal{E}_M that

$$4 \hat{\mathcal{I}}_{(1+\varphi)^{1/2}}^{(\mathcal{E}_M)}((\mathbb{1} + \varphi)\eta) = \hat{\mathcal{I}}_{\varphi}^{(\mathcal{E}_M)}(\eta)$$

for all $\eta \in B_{\mathcal{A}}(\mathcal{E}_M)_+$. Therefore

$$||| \hat{\mathcal{I}}_{(1+\varphi)^{1/2}}^{(\mathcal{E}_M)} ||| \leq 4^{-1} ||| \hat{\mathcal{I}}_{\varphi}^{(\mathcal{E}_M)} |||.$$

Combining these estimates one concludes that

$$||| \hat{\mathcal{I}}_{\varphi}^{(\mathcal{E})} ||| \leq (1 + \|\varphi\|_{\infty}) ||| \hat{\mathcal{I}}_{\varphi}^{(\mathcal{E}_M)} |||$$

for all $\varphi \in B_{\mathcal{A};\text{loc}}(\mathcal{E}_M)_+$.

Finally, replacing φ by $\tau\varphi$ with $\tau > 0$ and noting that $||| \hat{\mathcal{I}}_{\tau\varphi}^{(\mathcal{E})} ||| = \tau^2 ||| \hat{\mathcal{I}}_{\varphi}^{(\mathcal{E})} |||$ and $||| \hat{\mathcal{I}}_{\tau\varphi}^{(\mathcal{E}_M)} ||| = \tau^2 ||| \hat{\mathcal{I}}_{\varphi}^{(\mathcal{E}_M)} |||$ one deduces that

$$||| \hat{\mathcal{I}}_{\varphi}^{(\mathcal{E})} ||| \leq (1 + \tau \|\varphi\|_{\infty}) ||| \hat{\mathcal{I}}_{\varphi}^{(\mathcal{E}_M)} |||$$

for all $\varphi \in B_{\mathcal{A};\text{loc}}(\mathcal{E}_M)_+$. Therefore in the limit $\tau \rightarrow 0$ one has $||| \hat{\mathcal{I}}_{\varphi}^{(\mathcal{E})} ||| \leq ||| \hat{\mathcal{I}}_{\varphi}^{(\mathcal{E}_M)} |||$. Consequently $d^{(\mathcal{E}_M)}(A; B) \leq d^{(\mathcal{E})}(A; B)$ for all A, B .

The second step in the proof is to establish the converse inequalities. But this is a corollary of the small time asymptotics of the semigroups S and S^M established by Ariyoshi and Hino [AH05] and the order property of S and S^M given by Proposition 3.3.

First it follows from [AH05], Theorem 2.7, that

$$d^{(\mathcal{E})}(A; B)^2 = -\lim_{t \rightarrow 0} 4t \log(\mathbb{1}_A, S_t \mathbb{1}_B) \quad \text{and} \quad d^{(\mathcal{E}_M)}(A; B)^2 = -\lim_{t \rightarrow 0} 4t \log(\mathbb{1}_A, S_t^M \mathbb{1}_B)$$

for all A, B with $\mu(A), \mu(B) \in \langle 0, \infty \rangle$. (Note that our convention for the semigroup generator differs from that of [AH05] by a factor 2.)

Secondly, it follows from Proposition 3.3 that

$$(\mathbb{1}_A, S_t^M \mathbb{1}_B) \leq (\mathbb{1}_A, S_t \mathbb{1}_B)$$

for all $t > 0$. Therefore

$$d^{(\mathcal{E}_M)}(A; B)^2 = -\lim_{t \rightarrow 0} 4t \log(\mathbb{1}_A, S_t^M \mathbb{1}_B) \geq -\lim_{t \rightarrow 0} 4t \log(\mathbb{1}_A, S_t \mathbb{1}_B) = d^{(\mathcal{E})}(A; B)^2.$$

and one concludes that $d^{(\mathcal{E}_M)}(A; B) \geq d^{(\mathcal{E})}(A; B)$ for all A, B with $\mu(A), \mu(B) \in \langle 0, \infty \rangle$.

The proof of the second statement of the theorem, i.e. the equality $d^{(\mathcal{E}_m)}(A; B) = d^{(\mathcal{E})}(A; B)$, follows by repeating the foregoing argument with \mathcal{E} replaced by \mathcal{E}_m to conclude that $d^{(\mathcal{E}_m)}(A; B) = d^{(\mathcal{E}_M)}(A; B)$ and then combining this with the first statement of the theorem. In this case the Ariyoshi–Hino asymptotic estimate for S^m requires the strong locality of \mathcal{E}_m . Moreover, it uses the ordering for S^m and S^M given by Proposition 3.6 in place of the ordering of S and S^M given by Proposition 3.3. \square

Corollary 5.2 *Let \mathcal{E} and \mathcal{F} be strongly local inner regular forms with $\mathcal{E}_M \subseteq \mathcal{F}$. Assume $\mathcal{E}_m \leq \mathcal{F}$. Then*

$$d^{(\mathcal{F})}(A; B) = d^{(\mathcal{E})}(A; B) = d^{(\mathcal{E}_M)}(A; B) = d^{(\mathcal{F}_M)}(A; B)$$

for all measurable subsets A, B with $\mu(A), \mu(B) \in \langle 0, \infty \rangle$. Moreover, if \mathcal{E}_m is strongly local one also has $d^{(\mathcal{F}_m)}(A; B) = d^{(\mathcal{E})}(A; B)$.

Proof It follows from Theorem 5.1 applied to \mathcal{F} that

$$d^{(\mathcal{F})}(A; B) = d^{(\mathcal{F}_M)}(A; B)$$

for all A, B with $\mu(A), \mu(B) \in \langle 0, \infty \rangle$. But $\mathcal{E}_m \leq \mathcal{F} \leq \mathcal{E}_M$ by assumption and Corollary 4.2 establishes that this order relation is equivalent to the identity $\mathcal{F}_M = \mathcal{E}_M$. Therefore

$$d^{(\mathcal{F}_M)}(A; B) = d^{(\mathcal{E}_M)}(A; B) = d^{(\mathcal{E})}(A; B)$$

with the second identity following from Theorem 5.1. The last statement of the corollary follows by similar reasoning. \square

Note that although the distance $d^{(\mathcal{E})}(A; B)$ can in principle be infinite this is not the case if S is irreducible, i.e. if $(\mathbb{1}_A, S_t \mathbb{1}_B) > 0$ for all A, B with $\mu(A), \mu(B) \in \langle 0, \infty \rangle$ and $t > 0$. This follows because S satisfies the Davies–Gaffney bounds $(\mathbb{1}_A, S_t \mathbb{1}_B) \leq \mu(A)^{1/2} \mu(B)^{1/2} e^{-d^{(\mathcal{E})}(A; B)^2/4t}$ for all $t > 0$ (see [AH05] Theorem 4.1). On the other hand it is possible that $d^{(\mathcal{E})}$ is identically zero. In particular this happens if $D_0(\mathcal{E}) = \{0\}$, or $D_0(\mathcal{E}) = \{\lambda \mathbb{1}_X\}$ if $\mu(X) < \infty$. This follows because $\varphi \in D_0(\mathcal{E})$ requires the corresponding Radon measure μ_φ to be absolutely continuous with respect to μ and $d\mu_\varphi/d\mu \leq 1$. This condition typically fails for diffusions on fractals equipped with the Hausdorff measure.

Next we illustrate the foregoing results with the example of strictly elliptic forms. The equality of the various distances is a statement of independence of the geometry from the choice of boundary conditions for the corresponding diffusion process.

Example 5.3 Let \mathcal{E}_0 denote the Markovian form of Example 4.4 on $L_2(\Omega)$ where Ω is a domain in \mathbf{R}^d . Let \mathcal{E}_M and $\mathcal{E}_m (= (\mathcal{E}_M)_m)$ denote the corresponding extremal Dirichlet forms. Then $\mathcal{E}_m, \mathcal{E}_M$ and all Dirichlet forms \mathcal{E} with $\mathcal{E}_m \supseteq \mathcal{E} \supseteq \mathcal{E}_M$ are strongly local. Therefore $d^{(\mathcal{E}_m)}(A; B) = d^{(\mathcal{E})}(A; B) = d^{(\mathcal{E}_M)}(A; B)$ for all measurable subsets A, B with $0 < |A|, |B| < \infty$. Now we argue that if the coefficients c_{kl} are Lipschitz continuous then $d^{(\mathcal{E}_M)}(A; B)$ is the geodesic distance between the open subsets A and B corresponding to the metric C^{-1} .

First since \mathcal{E}_M is regular one can compute the set-theoretic distance with the nest of compact subsets of Ω . But

$$\mathcal{I}_\psi^{(\mathcal{E}_M)}(\xi) = \mathcal{E}_M(\psi, \xi \psi) - 2^{-1} \mathcal{E}_M(\xi, \psi^2) = \int_\Omega dx \xi(x) \Gamma(\psi)(x)$$

for $\xi, \psi \in D(\mathcal{E}_M)$ with compact support and with $\xi \geq 0$. Now if $\psi \in B_{\text{loc}}(\mathcal{E}_M)$ and $K \subset \Omega$ is compact one can choose $\hat{\psi} \in B(\mathcal{E}_M)$ such that $\psi|_K = \hat{\psi}|_K$. Then $\hat{\mathcal{I}}_\psi^{(\mathcal{E}_M)}(\xi) = \mathcal{I}_{\hat{\psi}}^{(\mathcal{E}_M)}(\xi)$ for all ξ with $\text{supp } \xi \subseteq K$. Therefore if $||| \hat{\mathcal{I}}_\psi^{(\mathcal{E}_M)} ||| \leq 1$ one has

$$\left| \int_K dx \xi(x) \Gamma(\psi)(x) \right| = \left| \int_K dx \xi(x) \Gamma(\hat{\psi})(x) \right| = |\mathcal{I}_\psi^{(\mathcal{E}_M)}(\xi)| = |\hat{\mathcal{I}}_\psi^{(\mathcal{E}_M)}(\xi)| \leq \|\xi\|_1.$$

Hence $\sup_{x \in K} |\Gamma(\psi)(x)| \leq 1$ uniformly for all K and all $\psi \in B_{\text{loc}}(\mathcal{E}_M)$. Thus $D_0(\mathcal{E}_M) = \{\psi \in W_{\text{loc}}^{1,\infty}(\Omega) : \|\Gamma(\psi)\|_\infty \leq 1\}$. Therefore $d^{(\mathcal{E}_M)}(A; B) = \inf_{x \in A, y \in B} d_C(x; y)$ where

$$d_C(x; y) = \sup\{\psi(x) - \psi(y) : \psi \in W_{\text{loc}}^{1,\infty}(\Omega), \|\Gamma(\psi)\|_\infty \leq 1\}.$$

But the latter expression is one of the well known characterizations of the geodesic distance.

The forms $\mathcal{E}_m, \mathcal{E}, \mathcal{E}_M$ are distinguished by different boundary conditions and the foregoing calculation establishes that the set-theoretic distance is independent of the boundary conditions. This conclusion also follows for the strongly local forms \mathcal{E}_F corresponding to Dirichlet boundary conditions on the closed subset F of the boundary and Neumann conditions on the complement $\partial\Omega \setminus F$ since $\mathcal{E}_m \leq \mathcal{E}_F \leq \mathcal{E}_M$.

Finally if Ω has a smooth boundary one can also define the forms $\mathcal{E}^{(\alpha)}$ with Robin boundary conditions by (19). These forms are local but not strongly local. Therefore the Ariyoshi–Hino definition of the set-theoretic distance is not applicable. Nevertheless one can deduce that the Robin semigroups have the same small time asymptotic behaviour as the Dirichlet and Neumann semigroups. This follows because

$$(\mathbb{1}_A, S_t^M \mathbb{1}_B) \leq (\mathbb{1}_A, S_t^{(\alpha)} \mathbb{1}_B) \leq (\mathbb{1}_A, S_t^m \mathbb{1}_B)$$

for all $A, B \subseteq \Omega$ with $0 < |A|, |B| < \infty$ and $t > 0$ by the discussion in Example 4.4. Then since $d^{(\mathcal{E}_m)}(A; B) = d^{(\mathcal{E}_M)}(A; B) (= d^{(\mathcal{E})}(A; B))$ it follows by the Ariyoshi–Hino asymptotic estimates for S^M and S^m that

$$d^{(\mathcal{E})}(A; B)^2 = -\lim_{t \rightarrow 0} 4t \log(\mathbb{1}_A, S_t^{(\alpha)} \mathbb{1}_B).$$

A similar argument is valid for the Robin semigroups defined for arbitrary domains in [AW03b] sinnce the sandwich estimate for the semigroups is established in [AW03a].

5.2 Uniqueness

In this subsection we consider the condition $\mathcal{E}_m = \mathcal{E}_M$. This condition has been extensively analysed as a criterion of Markov uniqueness of second-order elliptic operators on domains Ω in \mathbf{R}^d . In this latter setting it is equivalent to several rather different conditions (see [Ebe99], [RS11a] and [Rob13] for further details and references). In particular it known to be equivalent to two distinct types of capacity condition. The first result of this nature was due to Maz'ya (see [Maz85] Section 2.7 or [Ebe99] Theorem 3.6). Our next aim is to demonstrate that a characterization similar to that of Maz'ya can be established in the general Dirichlet form setting.

Recall that $B_{\text{cap}}(\mathcal{E})$ denotes the subspace of bounded functions in $D(\mathcal{E})$ whose supports have finite \mathcal{E} -capacity. If A is a subset of X with finite \mathcal{E} -capacity we then set $D_{\text{cap},A}(\mathcal{E}) = \{\psi \in D_{\text{cap}}(\mathcal{E}) : \text{supp } \psi \subseteq A\}$ and $B_{\text{cap},A}(\mathcal{E}) = D_{\text{cap},A}(\mathcal{E}) \cap L_\infty(X)$.

Proposition 5.4 *Let \mathcal{E} be a local, inner regular, Dirichlet form and $\mathcal{E}_m, \mathcal{E}_M$ the corresponding extremal forms. Consider the following conditions:*

- I. (I'.)** *for each subset A of X with finite \mathcal{E}_m -capacity there exists a sequence $\{\eta_n\}_{n \geq 1}$ of $\eta_n \in C_c(\mathcal{E}_M)$ with $0 \leq \eta_n \leq 1$ (of $\eta_n \in B(\mathcal{E}_M)$) such that*

$$\lim_{n \rightarrow \infty} \|(\mathbb{1}_X - \eta_n) \varphi\|_{D(\mathcal{E}_m)} = 0$$

for all $\varphi \in B_{\text{cap},A}(\mathcal{E}_m)$,

II. $\mathcal{E}_m = \mathcal{E}_M$.

Then $I \Rightarrow I' \Rightarrow II$. Moreover, if \mathcal{E}_M is strongly local then $II \Rightarrow I$ and the three conditions are equivalent.

Proof $I \Rightarrow I'$. This is evident.

$I' \Rightarrow II$. Fix $\varphi \in B_{\text{cap}}(\mathcal{E}_m)$ and set $A = \text{supp } \varphi$. Let η_n be the sequence in Condition I' corresponding to A and set $\varphi_n = \eta_n \varphi$. Since $B(\mathcal{E}_M)$ is an algebraic ideal of $B(\mathcal{E}_m)$, by Proposition 3.6, it follows that $\varphi_n \in B(\mathcal{E}_M)$. But $\mathcal{E}_m \supseteq \mathcal{E}_M$ by Theorem 3.1.III. Therefore Condition I' implies that the sequence φ_n is convergent with respect to the $D(\mathcal{E}_M)$ -graph norm. This establishes that $\varphi \in B(\mathcal{E}_M)$. Hence $B_{\text{cap}}(\mathcal{E}_m) \subseteq B(\mathcal{E}_M)$ and $\mathcal{E}_M(\varphi) = \mathcal{E}_m(\varphi)$ for all $\varphi \in B_{\text{cap}}(\mathcal{E}_m)$. But $B_{\text{cap}}(\mathcal{E}_m)$ is a core of \mathcal{E}_m by Proposition 2.6. Therefore $\mathcal{E}_m = \mathcal{E}_M$. $II \Rightarrow I$. We now assume \mathcal{E}_M . Hence \mathcal{E}_m is strongly local by Condition II. Since A has finite \mathcal{E}_m -capacity there is an $\eta \in D(\mathcal{E}_m)$ with $\eta = 1$ on A . Then $\eta \varphi = \varphi$ for all $\varphi \in B_{\text{cap},A}(\mathcal{E}_m)$. But $\eta \in D(\mathcal{E}_M)$, by Condition II, and it follows from the Dirichlet property that one may assume that $0 \leq \eta \leq 1$. Therefore one can choose a sequence $\eta_n \in C_c(\mathcal{E}_M)$ with $0 \leq \eta_n \leq 1$ such that $\|\eta_n - \eta\|_{D(\mathcal{E}_M)} \rightarrow 0$ as $n \rightarrow \infty$. In particular $\sup_n \mathcal{E}_M(\eta_n) < \infty$.

Next it follows that $\|(\eta_n - \eta)\varphi\|_2 \leq \|\eta_n - \eta\|_2 \|\varphi\|_\infty \rightarrow 0$ as $n \rightarrow \infty$ for all $\varphi \in B_{\text{cap},A}(\mathcal{E}_m)$. Moreover,

$$\mathcal{E}_m((\eta_n - \eta)\varphi) \leq 2 \mathcal{E}_m(\varphi) + 2 \mathcal{E}_m(\eta_n \varphi).$$

But it follows from the strong locality of \mathcal{E}_m , by a straightforward application of Theorem 5.2.1 in [BH91] that

$$\mathcal{E}_m(\eta_n \varphi) \leq 2 (\mathcal{E}_m)_{\eta_n^2}(\varphi) + 2 (\mathcal{E}_m)_{\varphi^2}(\eta_n).$$

Therefore one deduces that

$$\mathcal{E}_m((\eta_n - \eta)\varphi) \leq 6 \mathcal{E}_m(\varphi) + 4 \mathcal{E}_M(\eta_n) \|\varphi\|_\infty^2.$$

Consequently $\sup_n \mathcal{E}_m((\eta_n - \eta)\varphi) < \infty$. Hence there is a subsequence $(\eta_{n_k} - \eta)\varphi$ which is weakly convergent to zero in the Hilbert space $D(\mathcal{E}_m)$ equipped with the graph norm and with Cesaro mean strongly convergent to zero. Then replacing η_n by $\eta'_n = n^{-1} \sum_{k=1}^n \eta_{n_k}$ one deduces that $\|(\mathbb{1}_X - \eta'_n)\varphi\|_{D(\mathcal{E}_m)} = \|(\eta - \eta'_n)\varphi\|_{D(\mathcal{E}_m)} \rightarrow 0$. Therefore the sequence η'_n satisfies Condition I. \square

The situation is simpler if $\mu(X) < \infty$ and \mathcal{E} is strongly local. Then the identity function $\mathbb{1}_X$ is in the domain of the form \mathcal{E}_m and $\mathcal{E}_m(\mathbb{1}_X) = 0$. This follows by first noting that $\mathbb{1}_X \in L_2(X)$. Secondly, if $\xi, \varphi \in C_c(\mathcal{E})$ with $0 \leq \xi \leq 1$ and $\varphi = 1$ on $\text{supp } \xi$ then

$$\mathcal{E}_\xi(\varphi) = \mathcal{E}(\varphi, \xi\varphi) - 2^{-1}\mathcal{E}(\xi, \varphi^2) = \mathcal{E}(\varphi, \xi) - 2^{-1}\mathcal{E}(\xi, \varphi^2) = 0$$

by strong locality of \mathcal{E} . Therefore $\mathbb{1}_X \in D(\mathcal{E}_{m,Y;0})$ and $\mathcal{E}_{m,Y;0}(\mathbb{1}_X) = 0$ for all bounded open subsets Y of X . Then by the definition of \mathcal{E}_m one has $\mathbb{1}_X \in D(\mathcal{E}_m)$ and $\mathcal{E}_m(\mathbb{1}_X) = 0$.

Corollary 5.5 *Let \mathcal{E} be a strongly local, inner regular, Dirichlet form and $\mathcal{E}_m, \mathcal{E}_M$ the corresponding extremal forms. Assume $\mu(X) < \infty$. Then the following conditions are equivalent:*

- I. (I'.) *there exists a sequence $\{\eta_n\}_{n \geq 1}$ of $\eta_n \in C_c(\mathcal{E}_M)$ with $0 \leq \eta_n \leq 1$ (of $\eta_n \in B(\mathcal{E}_M)$) such that $\lim_{n \rightarrow \infty} \|\mathbb{1}_X - \eta_n\|_{D(\mathcal{E}_m)} = 0$,*
- II. $\mathcal{E}_m = \mathcal{E}_M$.

Proof $I \Rightarrow I'$. This is evident.

$I' \Rightarrow I$. One may first choose $\xi_n \in B_c(\mathcal{E}_M)$ such that $\|\eta_n - \xi_n\|_{D(\mathcal{E}_M)} \leq n^{-1}$ for all $n \geq 1$. Secondly one may choose $\zeta_n \in C_c(\mathcal{E}_M)$ such that $\|\xi_n - \zeta_n\|_{D(\mathcal{E}_M)} \leq n^{-1}$ for all $n \geq 1$ by inner regularity. Then one has $\|\mathbb{1}_X - \zeta_n\|_{D(\mathcal{E}_m)} \leq \|\mathbb{1}_X - \eta_n\|_{D(\mathcal{E}_m)} + 2n^{-1}$. Finally it follows from the Dirichlet property that the sequence $0 \vee \zeta_n \wedge 1$ satisfies Condition I.

$I \Rightarrow II$. Condition I implies that $\mathbb{1}_X \in B(\mathcal{E}_M)$. But $B(\mathcal{E}_M)$ is an algebraic ideal of $B(\mathcal{E}_m)$ by Proposition 3.6. Therefore $\varphi = \mathbb{1}_X \varphi \in B(\mathcal{E}_M)$ for all $\varphi \in B(\mathcal{E}_m)$. Consequently $B(\mathcal{E}_m) = B(\mathcal{E}_M)$ and $\mathcal{E}_m = \mathcal{E}_M$.

$II \Rightarrow I$. If $\mathcal{E}_m = \mathcal{E}_M$ then $C_c(\mathcal{E}_M)$ is a core of \mathcal{E}_m . Moreover, $\mathcal{E}_M = \mathcal{E}$ is strongly local. But $\mathbb{1}_X \in D(\mathcal{E}_m)$ by the preceding discussion. Therefore Condition I follows immediately from Proposition 5.4. \square

Mazya's criterion for uniqueness [Maz85], Section 2.7, was originally formulated for second-order elliptic operators in divergence form on \mathbf{R}^d . Then the capacity estimates of Proposition 5.4 and Corollary 5.5 give bounds on the possible growth of the coefficients at infinity which ensure that the 'boundary at infinity' is inaccessible to the corresponding diffusion. Complementary estimates have been given for operators on domains with boundaries in [RS11a] [RS11b]. We conclude by establishing these latter estimates for bounded domains.

First note that the \mathcal{E} -capacity of a subset $A \subseteq X$ is defined by

$$\text{cap}_{\mathcal{E}}(A) = \inf\{\|\varphi\|_{D(\mathcal{E})}^2 : \varphi \in D(\mathcal{E}), \varphi = 1 \text{ on } A\}$$

and one can restrict the infimum to φ with $0 \leq \varphi \leq 1$ by the Dirichlet property of \mathcal{E} . Secondly let Ω be a domain in X and \mathcal{E} a Dirichlet form on $L_2(\Omega)$. Then the capacity of subsets $A \subset \overline{\Omega}$ and in particular subsets of the boundary $\Gamma = \overline{\Omega} \setminus \Omega$ of Ω can be defined similarly. One sets

$$\text{cap}_{\mathcal{E}}(A) = \inf\{\|\varphi\|_{D(\mathcal{E})}^2 : \varphi \in D(\mathcal{E}) \text{ and there exists an open set } U \subset X \text{ with } A \subset U \text{ and } \varphi = 1 \text{ on } U \cap \Omega\}.$$

Now the second criterion for uniqueness is formulated in terms of the capacity of the boundary of Ω .

Proposition 5.6 *Let Ω be a bounded, open, connected subset of X with boundary Γ . Further let \mathcal{E} be a strongly local, inner regular, Dirichlet form on $L_2(\Omega)$ and $\mathcal{E}_m, \mathcal{E}_M$ the corresponding extremal forms. Then the following conditions are equivalent:*

- I. $\mathcal{E}_m = \mathcal{E}_M$,
- II. $\text{cap}_{\mathcal{E}_m}(\Gamma) = 0$.

Proof I \Rightarrow II. It follows from Corollary 5.5 that there exists a sequence η_n satisfying Condition I of the corollary. Since $\eta_n \in C_c(\Omega)$ it follows that $\Gamma \subset (\text{supp } \eta_n)^c$. Set $\varphi_n = \mathbf{1}_\Omega - \eta_n$. Then $0 \leq \varphi_n \leq 1$, $\varphi_n = 1$ on $(\text{supp } \eta_n)^c$ and $\|\varphi_n\|_{D(\mathcal{E}_m)} \rightarrow 0$ as $n \rightarrow \infty$. Therefore $\text{cap}_{\mathcal{E}_m}(\Gamma) = 0$.

II \Rightarrow I. Since $\text{cap}_{\mathcal{E}_m}(\Gamma) = 0$ there exists a sequence of open sets $U_n \subset X$ such that $\Gamma \subset U_n \cap \overline{\Omega}$ and a sequence of $\varphi_n \in D(\mathcal{E}_m)$ with $0 \leq \varphi_n \leq 1$, $\varphi_n = 1$ on $U_n \cap \Omega$ and $\|\varphi_n\|_{D(\mathcal{E}_m)} \rightarrow 0$ as $n \rightarrow \infty$. Now set $\eta_n = \mathbf{1}_\Omega - \varphi_n$. Then $\mathbf{1}_\Omega - \eta_n \in B(\mathcal{E}_m)$ and $\|\mathbf{1}_\Omega - \eta_n\|_{D(\mathcal{E}_m)} \rightarrow 0$ as $n \rightarrow \infty$. But $\text{supp } \eta_n \subseteq U_n^c \cap \Omega$. Therefore $\eta_n \in B(\mathcal{E}_M)$. Thus the sequence of η_n satisfies Condition I' of Corollary 5.5 which is equivalent to $\mathcal{E}_m = \mathcal{E}_M$. \square

The foregoing proof is based on the assumption that Ω is bounded. But boundedness is probably not essential for the conclusion of the proposition. The comparable result for the forms associated with elliptic differential operators on \mathbf{R}^d is valid for unbounded domains but the proof is considerably more complicated and depends on subadditivity of the capacity and localization arguments.

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